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A SIMPLE MAP BETWEEN FOKKER-PLANCK EQUATION AND ITS FRACTIONAL FORM

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Abstract

A simple map between Fokker-Planck Equation (FPE) and its fractional form (FFPE), which recently formulates to describe subdiffusive processes, has been suggested. This connection based on a relation between k-orders for moments of ordinary time domain of FPE and the moments associated with fractional time domain of FFPE . Two classes of special interest of FFPE has been considered to outline this map.

Keywords: *Fokker-Planck Equation, Fractional Fokker-Planck Equation, Anomalous Diffusion.*

INTRODUCTION

The Fokker-Planck equation (FPE) is currently one of the most important equation in the study of the dynamics process of nonequilibrium statistical systems. There are many models in biology, chemistry and physics whose analysis reduces to the problem of finding the solution to a Smoluchowki or FPE in one dimension [1-3]. Usually, the FPE can be derived following the Langevin approach that is starting from the stochastic equation of motion for the dynamic variable whose probability distribution is of interest.

Let us consider the FPE in one dimension in the form,

$$f(x,t) = \bar{f}_0(x) + \int_0^t d\tau (L_{fp}(x) f)(x,t) \tag{1}$$

where the operator *L(x)* is given by,

$$L_{fp}(x) = \frac{\partial}{\partial x}[D(x)\frac{\partial}{\partial x}] - \frac{\partial}{\partial x}[V(x)] \tag{2}$$

and *D(x)* and *V(x)* are referred to as the diffusion and drift coefficients respectively, their explicit forms vary considerably and depend on the particular application. The basic properties of Eq. (1) are exponential decay of the single modes in time,

$$T_n(t) = \exp(-\lambda_{n,1} t) \tag{3}$$

with $\lambda_{n,1}$ is the eigenvalue of the operator *L(x)*. Also Eq.(1) has the following formal solution,

$$f(x,t) = \exp(tL_{fp}(x))\ f_0(x) \tag{4}$$

where $f_0(x)$ is the initial condition. In the absence of the drift term i.e. $V(x) = 0$, the equation describes a Gaussian evolution as may be anticipated based on the central limit theorem. The mean-squared displacement $<X^2(t)>$ is proportional linearly with time t,

$$<X^2(t)> \; = Kt \tag{5}$$

where K is a constant related to the diffusion coefficient. Finally the stationary solution of Eq.(1),

$$f_{st}(x) = Lim_{t\to\infty} f(x,t) \tag{6}$$

is given by the Gibbs-Boltzmann distribution,

$$f_{st}(x) = \exp(V(x)/K_B T) \tag{7}$$

with $(K_B T)^{-1}$ denotes the Boltzmann factor.

In recent years, some interest has been focused on non-standard diffusion mechanism taking place in disordered system rather than Euclidean geometry [4]. Some typical examples of transport in fractured [5], phase-space motion in chaotic dynamics [6], and transport in heterogeneous media such as porous materials or gels [7] are some of the main instances where anomalous diffusion underlies transport process. Deviation from linearity, such that the mean squared displacement is proportional to t^γ, $\gamma \neq 1$, are the hallmarks of what is termed anomalous diffusion [8]. From the mathematical point of view, anomalous transport processes have posed new questions requiring different descriptions than normal diffusion. Fractional calculus provides a good material for this. Recently, the fractional Fokker-Planck equation (FFPE) has been introduced to describes anomalous types of relaxation and subdiffusion processes in the presence of an external field $F(x) = ((dV(x))/(dx))$. The FFPE describing subdiffusion behavior in an external field reads,

$$f^\gamma(x,t) = f_0(x) + {}_0D_t^{-\gamma}[L_{fp}(x) f](x,t) \tag{8}$$

where $f^\gamma(x,t)$ is the probability distribution function to find a particle (walker) at point x at time t in fractional time domain, and L(x) has the form given in expression (2). In Eq.(8), ${}_0D_t^{-\gamma}$ is the fractional Riemann-Liouville integral operator defined through [9],

$${}_0D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) d\tau \tag{9}$$

Eq. (8) can be rewritten in alternative form,

$$\frac{\partial}{\partial t} f^\gamma(x,t) = {}_0D_t^{1\gamma}[L_{fp}(x) f^\gamma](x,t) \tag{10}$$

Schnider and Wyss [10] have proposed a fractional Fick's equation, describing force-free anomalous diffusion of the type found in Eq. (8) when $D(x)$ and $V(x)$ are assumed to be constant,

$$\frac{\partial}{\partial t} f^\gamma(x,t) = {}_0D_t^{1-\gamma} k_\gamma \frac{\partial^2}{\partial x^2} f^\gamma(x,t) \tag{11}$$

where k_γ is a constant related to the diffusion coefficient in fractal geometry. It is obvious that for γ=1, Eq. (11) reduces directly to the standard diffusion equation. Moreover, it can be inferred from it through integration over $\int_{-\infty}^{\infty} dx x^2$, leading to

$$\frac{d}{dt} f^{\gamma}(x,t) = {}_0D_t^{1-\gamma} 2k_\gamma = 2k_\gamma \frac{t^{\gamma-1}}{\Gamma(\gamma)} \tag{12}$$

Upon solve Eq.(12), one can obtain the nonlinear growth of the mean squared displacement in the course of time as,

$$<X^2(t)> = \frac{2k_\gamma}{\Gamma(\gamma+1)} t^{\gamma} \tag{13}$$

Many works have focused on the derivation of the FFPE using different approaches as well as the domain of its validity. For instance, in Ref. [11] the FFPE was derived from a generalized continuous time random walk(CTRW), that includes space dependent jump probabilities which are the result of an external field *F(x)*. In additional, other approaches for the derivation of FFPE have been introduced in [11, see for instance Refs. therein]. Completely different approach has been introduced in [12] for the derivation of FFPE where we used the comb-like model as a background media for the random walk. Also in this work [12], a direct connection between fractal structure such as comb-like model and fractional dynamics has been established. More recently, based on a relationship between the input and out put of what is called fractal transmitted system, the FFPE with variable coefficients involving an external potential term has been derived in [13,14]. The main concern of this work is to construct a simple relation between the standard FPE and its fractional counterpart. This suggests relation will map the moments associated with FPE and the moments corresponding to the FFPE.

The MAP BETWEEN FPE AND FFPE

Consider Eqs.(1) and (8) associated with the following initial and boundary conditions are given by,

$$f^{\gamma,1}(x,t) = f_0(x) = \delta(x-x_0) \tag{14}$$

$$Lim_{x\to\pm\infty} f^{\gamma,1}(x,t) = 0 \tag{15}$$

Therefore Eqs.(1) and (8) read as,

$$\int_{-\infty}^{\infty} f^{\gamma,1}(x,t) dx = 1 \tag{16}$$

where $f^1(x,t)$ and $f^{\gamma}(x,t)$ are referred to the solution of FPE and FFPE respectively. In order to obtain the map $f^1(x,t)$) and $f^{\gamma}(x,t)$, multiply Eqs. (1) and (8) by x^k and integrating over $x\varepsilon[-\infty,\infty]$, with making use the moments definition,

$$\int_{-\infty}^{\infty} x^k f^{1,\gamma}(x,t) dx = M_k^{1,\gamma}(t) \qquad k = 1,2... \tag{17}$$

Therefore Eqs.(1) and (8) read as,

$$M_k^1(t) = M_k(0) + \int_0^t d\tau [A(E^{\pm}) M_k^1](x,t) \tag{18}$$

and

$$M_k^{\gamma}(t) = M_k(0) + \int_0^t d\tau [A(E^{\pm}) M_k^{\gamma}](x,t) \tag{19}$$

where $A(E^{\pm})$ is an operator depends only on the shift operator $(E^{\pm})$ defined through [15],

$$E^{\pm n} M_k(t) = M_{k\pm n}(t) \tag{20}$$

To observe the relation between Eqs.(18) and (19), one can use the Laplace and Mellin transforms defined by,

$$M(p) = L\{M(t), p\} = \int_0^{\infty} \exp(-pt) M(t) dt \tag{21}$$

and

$$M(s) = \overline{M}\{M(t), s\} = \int_0^{\infty} t^{s-1} M(t) dt \tag{22}$$

respectively, also one further the relation,

$$\overline{M}\{M(t), s\} = \frac{1}{\Gamma(1-s)} \overline{M}\{L\{M(t), p\}, 1-s\} \tag{23}$$

connecting Laplace and Mellin transforms. By employing Laplace transformation to Eqs.(18) and (19), one obtains

$$[A(E^{\pm n}) M_k^1](p) = p M_k^1(p) - M_k(0) \tag{24}$$

and

$$[A(E^{\pm n}) M_k^{\gamma}](p) = p^{\gamma} M_k^{\gamma}(p) - p^{\gamma-1} M_k(0) \tag{25}$$

From the above expressions (24) and (25), it is obvious that the relation taking the form,

$$M_k^{\gamma}(p) = p^{\gamma-1} M_k^1(p^{\gamma}) \tag{26}$$

Making use (23), formula (26) in Mellin domain can be written as,

$$M_k^{\gamma}(s) = (1/\gamma) \frac{\Gamma(1-(s/\gamma))}{\Gamma(1-s)} M_k^1(s/\gamma) \tag{27}$$

It is easily to see that, formula (27) gives a very simple relation between moments in time domain and fractional one. Once the *k*-order moments evaluate, one can directly obtain the corresponding subdiffusive moments. The task now is the calculation of an infinite hierarchy of coupled equations for the consecutive moments in ordinary time domain. Two different methods will presents here to obtain the *k*-order moments for FPE, namely, operator method [16], and moment expansion technique [17]. The validity and reliability of suggest connection (27) is tested by its application to some different forms of FPE. Two different examples are chosen for the interesting nature of the solution, but they are by no means claimed to be unique, there may be other similar examples perhaps more interesting. Also it is worth noting that, in the relaxation process problem in diffusion and subdiffusion domain described by FPE and FFPE respectively, the first and the second moments would be of interest from which one can known the statistical behavior of the system [16-18].

APPLICATION

Example (1):

In the first example, we apply the operator method to solve the moment equation of the model study the genetic population,

$$\frac{\partial}{\partial t} f(x,t) = \beta \frac{\partial^2}{\partial x^2}[xf(x,t)] - \alpha_0 \frac{\partial}{\partial x}[xf(x,t)] \tag{28}$$

where β and α_0 are constants. Multiplying Eq.(28) by x^{k} and integrating over x ε [-∞,∞], the moment equation reads,

$$\frac{d}{dt} M_k(t) = [\hat{A} + \hat{B}] M_k(t) \tag{29}$$

With,

$$\begin{aligned} \hat{A} &= \alpha_0 k, \\ \hat{B} &= \beta k(k-1)E^{-1} \end{aligned} \tag{30}$$

E^{-1} defined as the shift operator and given by,

$$E^{-1} M_k(t) = M_{k-1}(t) \tag{31}$$

The formal solution of (29) directly reads as,

$$M_k(t) = \exp(t[\hat{A} + \hat{B}]) M_k(0) \tag{32}$$

By making use the decomposition rule introduced in [19], one has

$$\exp(t[\hat{A} + \hat{B}]) = \exp(t\hat{A}) \exp(f(\alpha_0 t)\hat{B} \tag{33}$$

Where the commutation relations are given by,

$$\begin{aligned} [\hat{A} + \hat{B}]) &= \alpha_0 \\ f(\alpha_0 t) &= \frac{1 - \exp(-\alpha_0 t)}{\alpha_0 t} \end{aligned} \tag{34}$$

Therefore, the formal solution can be written in the following form,

$$M_k(t) = \exp(t\hat{A})\{x_0^k + \sum_{n=1}^{\infty} \frac{(1-\exp(-\alpha_o t))^n}{\alpha_0^n n!} (\beta k(k-1))^n M_{k-n}(0)\} \tag{35}$$

Eq. (35) gives a closed form for the moments, so the different orders of the moments can be directly evaluated. For instance, the first moment at *k*=1, is

$$M_1(t) = x_0 \exp(\alpha_0 t) \tag{36}$$

And the second order i.e. *k*=2, reads

$$M_2(t) = (x_0^2 + \frac{2\beta x_0}{\alpha_0}) \exp(2\alpha_0 t) - \frac{2\beta x_0}{\alpha_0} \exp(\alpha_0 t) \tag{37}$$

To obtain the corresponding fractional form for $M_1(t)$, Eq.(36) in Laplace domain becomes,

$$M_1(t) = \frac{x_0}{(p - \alpha_0)} \tag{38}$$

Using the connection between Laplace and Mellin (23) for (38), yields

$$M_1^{\gamma}(s) = \frac{\Gamma(1-(s/\gamma))}{\Gamma(1-s)}(\alpha_0^{1/\gamma})^{-s} \tag{39}$$

Expression (39) can be expressed in the time domain via comparing it with the definition of *H*-Function [20, 21] as

$$M_1^{\gamma}(t) = (x_0/\gamma)H_{12}^{11}\left((-\alpha_0)^{1/\gamma}t\left|\begin{smallmatrix}(0,1/\gamma)\\(0,1/\gamma)(0,1)\end{smallmatrix}\right.\right) \tag{40}$$

Here

$$H_{pq}^{mn}(Z) = H_{pq}^{mn}\left(Z\left|\begin{smallmatrix}(a_j,\alpha_j)_{j=1\dots p}\\(b_j,\beta_j)_{j=1\dots q}\end{smallmatrix}\right.\right) \tag{41}$$

denotes the *H*-function, and can be represented by contour integration as,

$$H_{pq}^{mn}(Z) = \frac{1}{2\pi i}\int\frac{A(s)B(s)}{C(s)D(s)}ds \tag{42}$$

With

$$\begin{aligned}
A(s) &= \prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\\
B(s) &= \prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\\
C(s) &= \prod_{j=1}^{m}\Gamma(1-b_j-\beta_j s)\\
D(s) &= \prod_{j=1}^{n}\Gamma(a_j-\alpha_j s)
\end{aligned} \tag{43}$$

H-function can be written in a series expansion such as,

$$M_1^{\gamma}(t) = \left\{x_0\sum_{r=0}^{\infty}\frac{(-\alpha_o)^r t^{r\gamma}}{\Gamma(\gamma r+1)}\right\} = E_{\gamma}(\alpha_0 t^{\gamma}) \tag{44}$$

$E_{\gamma}(Z)$ denotes the Mittage-Leffler function[22]. At γ =1, Eq.(44) reduces directly to (36). The asymptotic behavior for (44) interpolates between a power law decay for large time and for short time the relaxation is a stretched exponential. In a similar way, the second moment takes the form,

$$M_2^{\gamma}(t) = (x_0^2+\frac{2\beta x_0}{\alpha_0})E_{\gamma}(2\alpha_0 t^{\gamma}) - \frac{2\beta x_0}{\alpha_0}E_{\gamma}(\alpha_0 t^{\gamma}) \tag{45}$$

Example (2):

The second example is so-called Kimura model [22], describing population genetics,

$$\frac{\partial}{\partial t}f(x,t) = \frac{\partial^2}{\partial x^2}[x^2 f(x,t)] - (1-c)\frac{\partial}{\partial x}[xf(x,t)] \tag{46}$$

where c is a constant. We will solve this example using the expansion moment technique introduced in [18]. Let us assume the solution of Eq. (46) has the series form,

$$f(x,t) = \sum_{r=0}^{\infty}f_r(x)\frac{t^r}{r!} \tag{47}$$

Multiplying above expression by x^k and integrating over $x\,\varepsilon$ [-∞,∞], with the use of moment definition, one has

$$M_k(t) = \sum_{r=0}^{\infty} F_{k,r} \frac{t^r}{r!}, \qquad k, r = 0,1,2... \tag{48}$$

with

$$F_{k,r} = \int_{-\infty}^{\infty} x^k f_r(x) dx = < x^k >_r \tag{49}$$

Inserting solution (47) into (46) and multiplying by x^k and then integrating over $x\,\varepsilon$ [-∞,∞] with making use of (49), one finds

$$\begin{aligned} F_{1,r} &= x_0 (1 - c\tau)^r \\ F_{2,r} &= 2^r x_0 (2 - c\tau)^r \end{aligned} \tag{50}$$

where τ is a parameter having the dimension of time. In connection with Eq.(48), the first two moments becomes

$$\begin{aligned} M_1(t) &= x_0 \exp(\frac{1 - c\tau}{\tau})t \\ M_2(t) &= x_0^2 \exp[2(\frac{2 - c\tau}{\tau})t] \end{aligned} \tag{51}$$

or in general form, the moment expression will take the following form,

$$M_k(t) = x_0^k \exp[k(\frac{k - c\tau}{\tau})t] \tag{52}$$

In the same manner, via map (27) the corresponding fractional moments can directly evaluate in the following form

$$M_1^{\gamma}(t) = x_0 E_{\gamma}(-\theta t^{\gamma}) \tag{53}$$

with θ=(c-(1/τ)), the asymptotic behavior for (53) for t→∞, is

$$M_1^{\gamma}(t) \approx \frac{x_0 (\theta t)^{-\gamma}}{\Gamma(1 - \gamma)} \tag{54}$$

and for small t becomes

$$M_1^{\gamma}(t) \approx x_0 \exp(\frac{-\theta t^{\gamma}}{\Gamma(1 + \gamma)}) \tag{55}$$

In general, the *k*-orders in fractional time domain will be

$$M_k^{\gamma}(t) = x_0^k E_{\gamma}(-k\theta t^{\gamma}) \tag{56}$$

CONCLUSION

Anomalous random motions and related transport phenomena are ubiquitous in nature. In these phenomena the laws of normal diffusion (ordinary Brownian motion) are altered, e.g., the mean squared no longer increases linearly in time, but instead grows slower (subdiffusion) than the linear function. On other hand, the properties of these scaling media are often described by fractal functions and their space-time evolution. However, such functions contain hierarchies of singularities and are typically non-differentiable. Thus, the understanding of such phenomena comes about through the development and implementation of alternate modeling strategies that do not explicitly include the usual differential equations of motion. The use of differential equations with partial fractional derivative is a perspective way for describing such processes. The main result of this paper concerns a transformation of ordinary Gaussian diffusion into fractional diffusion, i.e. one can present a simple map connected between the k-orders of moments for FPE and FFPE. The method summarize as follows:

(i) By using operator method or moment expansion technique, one can easily obtain the k-orders of moments of FPE.

(ii) Via expression (27), one can be able to calculate the k-orders associated with FFPE.

Two interesting examples have been introduced in details to outline our approach.

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