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TRANSIENT NEUTRON TRANSPORT IN SEMI-INFINITE MEDIA FOR PURE-TRIPLET SCATTERING

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Abstract

The transient neutron transport in a semi-infinite medium with pure-triplet scattering is presented. Case's eigenfunctions for this problem can be obtained for this high order anisotropic scattering and orthogonality relations of these eigenfunctions can be derived mathematically. The reflectivity at the boundary, radiant energy and net heat flux are computed for specular-reflecting boundary with angular-dependent externally-incident flux. For the sake of comparison, we use two different weight functions in our calculations.

Keywords: *Transient neutron transport, Semi-infinite medium, Pure-triplet scattering, Specular-reflecting boundary*.

INTRODUCTION

The time-dependent transport equation has important applications in many fields, such as in reactor physics and astrophysics [1], [2], [3]. Many papers have been published which solved the one-dimensional time-dependent transport equation in an infinite and semi-infinite slabs [4] - [9]. Several methods were proposed to solve the time-dependent transport equation in finite slabs such as the diamond difference implicit trapezoidal [10], the analytical constant nodal [11], the multiple-collision [12], the discrete ordinates and semi-analytical numerical [13], [14], methods. Also, El-Wakil et al. have been solved the time-dependent radiative transfer problem by using the Pomraning-Eddington approximation for both linear anisotropic scattering [15] and Rayleigh scattering [16]. Su [17] used the spherical harmonic P_N approximation for Marshak wave problem [5] with an isotropic incident radiation flux to yield exact radiation energy density and material temperature at the boundary for all times when the Mark boundary conditions is used [19].

In this work, the monoenergetic time-dependent neutron transport equation in a semi-infinite medium is proposed. Here the scattering of neutrons from nucleus is definite as pure-triplet. The medium is considered to have specular-reflecting boundary with angular-dependent externally-incident flux. The problem is transformed into a stationary-like problem and then Case's eigenfunctions can be obtained. The solution gives the neutron distribution density function which is then used to calculate some interesting physical quantities such as reflectivity, radiant energy and net heat flux. A weight function is introduced to force the assumed boundary conditions to be fulfilled. For the sake of comparison, two different forms of weight functions are used.

ANALYSIS

Consider the monoenergetic time-dependent, one speed neutron transport equation with anisotropic scattering as [20], [21]

$$\left[\frac{1}{v}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial x}+\sigma(x,t)\right]N\left(x,\mu,t\right)=\frac{\sigma_s(x,t)}{2}\int_{-1}^{1}P\left(\mu,\mu'\right)N\left(x,\mu',t\right)d\mu' \qquad \textbf{(1)}$$

$$0\le x<\infty, \quad -1<\mu\le 1$$

where

- $N(x, \mu, t)$ is the neutron density distribution function with temporal variable t, geometrical space variable x, and the angular variable μ (the direction cosine of the transported neutrons).
- v is the speed of the neutrons under consideration.
- $\sigma(x,t)$ is the total cross-section (which is the total probability of a neutron interaction), $\sigma(x,t) = \sigma_a (x,t) + \sigma_s (x,t)$.
- $\sigma_a (x,t)$ is the absorption cross-section (which is the probability that a neutron at position x and time t will be absorbed).
- $\sigma_s (x,t)$ is the scattering cross-section (which is the probability that a neutron at position x and time t will have a collision and scatter).
- $P(\mu, \mu')$ is the anisotropic scattering phase function which can be expanded by terms of Legendre polynomials as [22]

$$P\left(\mu,\mu'\right)=\sum_{n=0}^{\infty}a_nP_n\left(\mu\right)P_n\left(\mu'\right)$$

$$=1+a_1\mu\mu'+a_2P_2(\mu)P_2(\mu')+a_3P_3(\mu)P_3(\mu')+\cdots \qquad \textbf{(2)}$$

with $P_n(\mu)$ is the Legendre polynomial functions.

Here a_n can be called the anisotropy scattering coefficients with $a_0 = 1$. The first term of this expansion is called isotropic scattering ($a_1 = a_2 = a_3 = 0$). The probability of particle scattering is equal for all directions in isotropic scattering. The second term in this expansion corresponds to the linearly anisotropic scattering ($a_1 \neq 0$, $a_2 = a_3 = 0$). The third term corresponds to the quadratic scattering (Rayleigh scattering $a_1 = 0$, $a_2 = 0.5$, $a_3 = 0$). The fourth term corresponds to the triplet scattering ($a_1 = a_2 = 0$, $a_3 \neq 0$). Here, the scattering of neutrons from nucleus is considered as pure-triplet.

It is convenient to write Eq.(1) in terms of the optical depth space variable

$$\tau(x)=\int_0^x\sigma(x)dx \qquad \textbf{(3)}$$

to become

$$\left[\frac{1}{u}\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\tau}+1\right]\Psi\left(\tau,\mu,t\right)=\frac{\omega}{2}\int_{-1}^{1}P\left(\mu,\mu'\right)\Psi\left(\tau,\mu',t\right)d\mu' \qquad \textbf{(4)}$$

where

$$\Psi\left(\tau,\mu,t\right)\equiv N\left(x,\mu,t\right) \qquad \textbf{(5.a)}$$

$$\omega=\sigma_s/\sigma \qquad \text{and} \qquad u=v\sigma \qquad \textbf{(5.b)}$$

Equation (4) is assumed to subject to the specular-reflecting boundary and initial conditions

$$\Psi(0,\mu,t)=\Gamma(\mu)+\rho^s\Psi(0,-\mu,t), \quad \text{at} \quad t=0 \ , \qquad \textbf{(6)}$$

$$\lim_{\tau\to\infty}\Psi(\tau,-\mu,t)=0 \qquad \textbf{(7)}$$

where $\Gamma(\mu)$ is the angular-dependent externally-incident flux, and ρ^s is the specular reflectivity of this boundary.

For solving Eq.(4), we use the transformation [7] - [9], [15], [16]

$$\xi=\tau+ut \qquad \textbf{(8)}$$

In terms of ξ, Eq.(4) becomes

$$\left[(1+\mu)\frac{\partial}{\partial\xi}+1\right]\Psi(\xi,\mu)=\frac{\omega}{2}\int_{-1}^{1}P(\mu,\mu')\Psi(\xi,\mu')\,d\mu' \qquad \textbf{(9)}$$

For pure-triplet scattering, we get Eq.(9) as

$$\left[(1+\mu)\frac{\partial}{\partial\xi}+1\right]\Psi(\xi,\mu)=\frac{\omega}{2}\int_{-1}^{1}\left[1+a_3P_3(\mu)P_3(\mu')\right]\Psi(\xi,\mu')\,d\mu' \qquad \textbf{(10)}$$

It well known that the transport equation of the type given by Eq.(10) admit separable exponential solutions of the form [2]

$$\Psi(\xi,\mu)=A\phi(\mu,\nu)\exp(-\nu\xi) \qquad \textbf{(11)}$$

where $\phi(\mu,\nu)$ is a normalized function and A is the normalization constant to be determined.

Using of Eq.(11) in Eq.(10) yields

$$(b-\nu\mu)\phi(\mu,\nu)=\frac{\omega}{2}\left[h_0(\nu)+a_3P_3(\mu)h_3(\nu)\right] \qquad \textbf{(12)}$$

with

$$h_i(\nu)=\int_{-1}^{1}P_i(\mu)\phi(\mu,\nu)\,d\mu \quad \text{and} \quad b=1-\nu \qquad \textbf{(13)}$$

Integrating Eq.(12) over $\mu\in[-1, 1]$ gives

$$(b-\omega)h_0(\nu)-\nu h_1(\nu)=0 \qquad \textbf{(14)}$$

Multiplying Eq.(12) by μ and μ^2 and integrate over $\mu\in[-1, 1]$, one gets, respectively

$$bh_1(\nu)-\frac{\nu}{3}\left[h_0(\nu)+2h_2(\nu)\right]=0 \qquad \textbf{(15)}$$

$$\frac{b}{3}\left[h_0(\nu)+2h_2(\nu)\right]-\frac{\nu}{5}\left[3h_1(\nu)+2h_3(\nu)\right]=\frac{\omega}{3}h_0(\nu) \qquad \textbf{(16)}$$

Dividing Eq.(12) by $(b-\nu\mu)$ and integrate over $\mu\in[-1, 1]$ we get

$$\left[1-\frac{\omega}{\nu}q_0\left(\frac{b}{\nu}\right)\right]h_0(\nu)-\frac{a_3\omega}{\nu}q_3\left(\frac{b}{\nu}\right)h_3(\nu)=0 \qquad \textbf{(17)}$$

where we have defined the $n^{\underline{th}}$ order Legendre function of second kind, $q_n(y)$, as

$$q_n(y)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(\mu)}{y-\mu}\,d\mu \qquad \textbf{(18)}$$

with

$$q_0\left(\frac{b}{\nu}\right)=\frac{1}{2}\ln\left(\frac{b+\nu}{b-\nu}\right) \qquad \textbf{(19.a)}$$

$$q_3\left(\frac{b}{\nu}\right)=\frac{b}{4\nu^3}\left(5b^2-3\nu^2\right)\ln\left(\frac{b+\nu}{b-\nu}\right)-\frac{5b^2}{2\nu^2}+\frac{2}{3} \qquad \textbf{(19.b)}$$

Eqs.(14)-(17) constitute four linear homogeneous algebraic equations for the four unknowns h_0, h_1, h_2 and h_3. The corresponding vanishing of the coefficient determinant gives the transcendental equation (characteristic equation) satisfied by ν, as

$$\begin{vmatrix} b-\omega & -\nu & 0 & 0 \\ -\dfrac{\nu}{3} & b & -\dfrac{2\nu}{3} & 0 \\ \dfrac{b-\omega}{3} & -\dfrac{3\nu}{5} & \dfrac{2b}{3} & -\dfrac{2\nu}{5} \\ 1-\dfrac{\omega}{\nu}q_0 & 0 & 0 & \dfrac{-a_3\omega}{\nu}q_3 \end{vmatrix} = 0 \quad \textbf{(20)}$$

Now for normalized $\phi(\mu,\nu)$ [i.e. $h_0(\nu) = 1$], we have

$$h_1(\nu) = \frac{1}{\nu}(b-\omega) \quad \textbf{(21)}$$

$$h_2(\nu) = \frac{1}{2}\left[1 - \frac{3b}{\nu^2}(b-\omega)\right] \quad \textbf{(22)}$$

and

$$h_3(\nu) = \frac{5b}{6\nu} - \left(\frac{5b^2}{2\nu^3} + \frac{2}{3\nu}\right)(b-\omega) \quad \textbf{(23)}$$

Therefore, from Eq.(12) we obtain

$$\phi(\mu,\nu) = \frac{\omega/2}{(b-\nu\mu)}\left\{1 + a_3P_3(\mu)\left[\frac{5b}{6\nu} - (b-\omega)\left(\frac{5b^2}{2\nu^3} + \frac{2}{3\nu}\right)\right]\right\} \quad \textbf{(24)}$$

and hence we obtain the solution in the form

$$\Psi(\tau,\mu,t) = \frac{\omega A e^{-\nu ut}}{2(b-\nu\mu)}\left\{1 + a_3P_3(\mu)\left[\frac{5b}{6\nu} - (b-\omega)\left(\frac{5b^2}{2\nu^3} + \frac{2}{3\nu}\right)\right]\right\}\exp(-\nu\tau) \quad \textbf{(25)}$$

The constant A can be determined by introducing a weight function $W(\mu)$ in order to force the boundary conditions Eq.(6) to be fulfilled, as

$$\int_0^1 d\mu W(\mu)\left[\Psi(0,\mu,t) - \Gamma(\mu) - \rho^s\Psi(0,-\mu,t)\right] = 0 \quad \textbf{(26)}$$

this can give

$$A = \left(\frac{2}{\omega}\right)\frac{I_0}{I_+ - \rho^s I_-} \quad \textbf{(27)}$$

where

$$I_0 = \int_0^1 W(\mu)\Gamma(\mu)d\mu \quad \text{and} \quad I_\pm = \int_0^1 W(\mu)J(\pm\mu)d\mu \quad \textbf{(28)}$$

with

$$J(\pm\mu) = \frac{1}{(b\mp\nu\mu)}\left\{1 + a_3P_3(\mu)\left[\frac{5b}{6\nu} - (b-\omega)\left(\frac{5b^2}{2\nu^3} + \frac{2}{3\nu}\right)\right]\right\} \quad \textbf{(29)}$$

The solution is, then given by

$$\Psi(\tau,\mu,t) = \frac{I_0 e^{-\nu ut}}{I_+ - \rho^s I_-}J(\mu)\exp(-\nu\tau) \quad \textbf{(30)}$$

NUMERICAL RESULTS

Once, we have obtained the solution of the neutron transport problem in a semi-infinite medium with pure-triplet scattering, (31), we can calculate the reflectivity at the boundary, radiant energy and net heat flux of the transported neutrons, respectively, from

$$R = \int_0^1 \mu \Psi(0, -\mu, t)\, d\mu, \quad \textbf{(31)}$$

$$E(\tau, t) = \int_{-1}^{1} \Psi(\tau, \mu, t)\, d\mu, \quad \textbf{(32)}$$

$$F(\tau, t) = \int_{-1}^{1} \mu \Psi(\tau, \mu, t)\, d\mu \quad \textbf{(33)}$$

The angular-dependent externally-incident flux $\Gamma(\mu)$ is assumed to have the form

$$\Gamma(\mu) = \mu^{\ell}, \quad \ell = 0, 1, 2, \cdots \quad \textbf{(34)}$$

Two different weight functions are suggested to do the calculations, namely [23], [24]

$$W_1(\mu) = \mu, \quad \textbf{(35.a)}$$

$$W_2(\mu) = \frac{\sqrt{3}}{2}\mu\left(1 + \frac{3}{2}\mu\right) \quad \textbf{(35.a)}$$

The numerical results for reflectivity R, radiant energy $E(\tau,t)$ and net flux $F(\tau,t)$ are represented in graphical form. The plotted figures are done by using the two weight functions $W_1(\mu)$ and $W_2(\mu)$. However, the three dimensional graphs are plotted for $W_1(\mu)$. From the plotted figures, one can see a good agreement for the calculated results.

FIGURES CAPTIONS

1. Figs.(1) represent the reflectivity R versus the time t (s) for:
(a) isotropic scattering ($a_3 = 0$), $\ell = \rho^s = 0$ and $\omega = 0.9$,
(b) pure-triplet scattering ($a_3 = 0.5$), $\ell = 1$, $\rho^s = 0.5$ & $\omega = 0.7$.

2. Figs.(2) give three dimensional graphs for the reflectivity R for pure-triplet scattering ($a_3 = 0.5$) versus:
(a) {ℓ and ρ^s } for $\omega = 0.6$ and at time = 0.1 s,
(b) {t and ρ^s } for $\omega = 0.8$ and $\ell = 1$.

3. Figs.(3) show the radiant energy $E(\tau, t)$ of the neutrons versus:
(a) time t (s) for isotropic scattering $a_3 = 0$ and $\ell = \rho^s = 0$, $\omega = 0.9$ & $\tau = 1$ *mfp*,
(b) optical distance τ (*mfp*) inside the medium for isotropic scattering $a_3 = 0$ and $\ell = 0$, $\rho^s = 0.25$, $t = 0.1$ s & $\omega = 0.9$,
(c) optical distance τ (*mfp*) inside the medium for pure-triplet scattering $a_3 = 0.5$ and $\ell = 1$, $\rho^s = 0.5$, $t = 3$ s & $\omega = 0.7$.

4. Figs.(4) represent the net flux $F(\tau, t)$ of the neutrons for the same terms and values of Figs.(3).

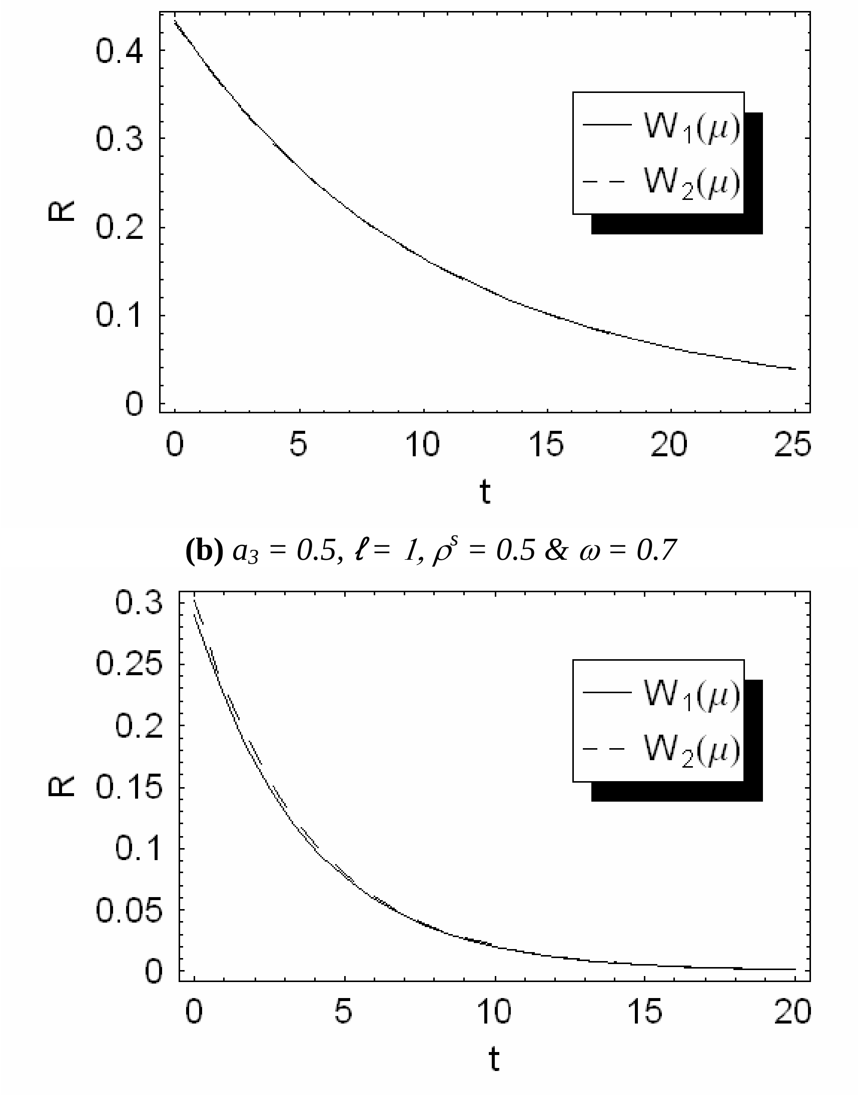
5. Figs.(5) give three dimensional graphs for the radiant energy $E(\tau, t)$ versus {τ and t}:
(a) for pure-triplet scattering $a_3 = 0.5$ and $\ell = \rho^s = 0$ & $\omega = 0.8$,

(b) for isotropic scattering $a_3 = 0$ and $\ell=1$, $\rho^s = 0.25$ & $\omega = 0.9$.

6. Figs.(6) give three dimensional graphs for the net flux $F(\tau, t)$ versus {τ and t} for the same terms and values of Figs.(5).

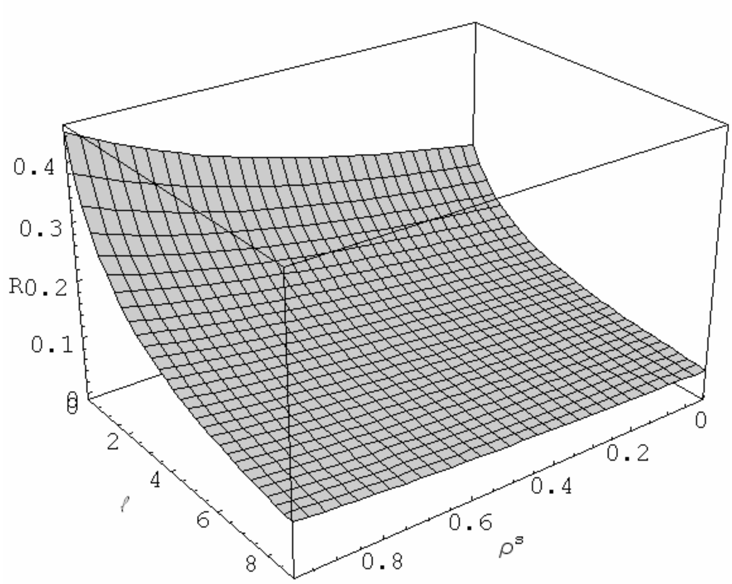
(a) $a_3 = 0, \ell = \rho^s = 0$ & $\omega = 0.9$

(b) $a_3 = 0.5, \ell = 1, \rho^s = 0.5$ & $\omega = 0.7$

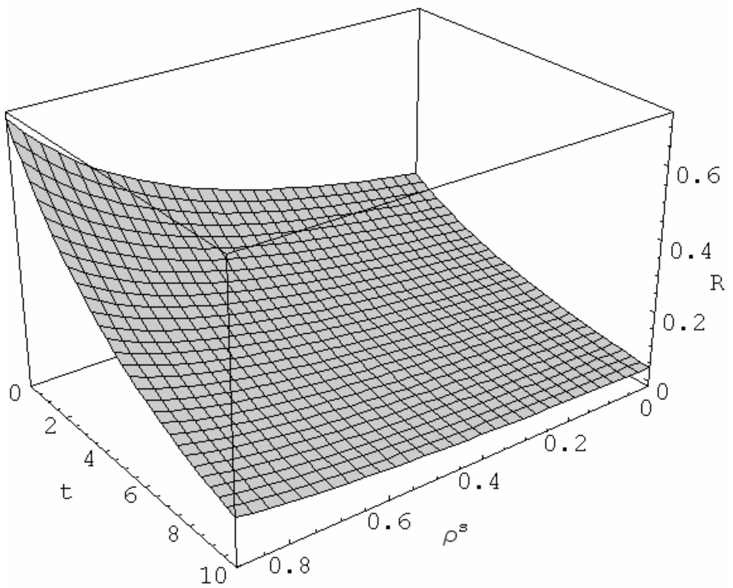


Figs.(1): The reflectivity R versus time t for isotropic and pure-triplet scattering

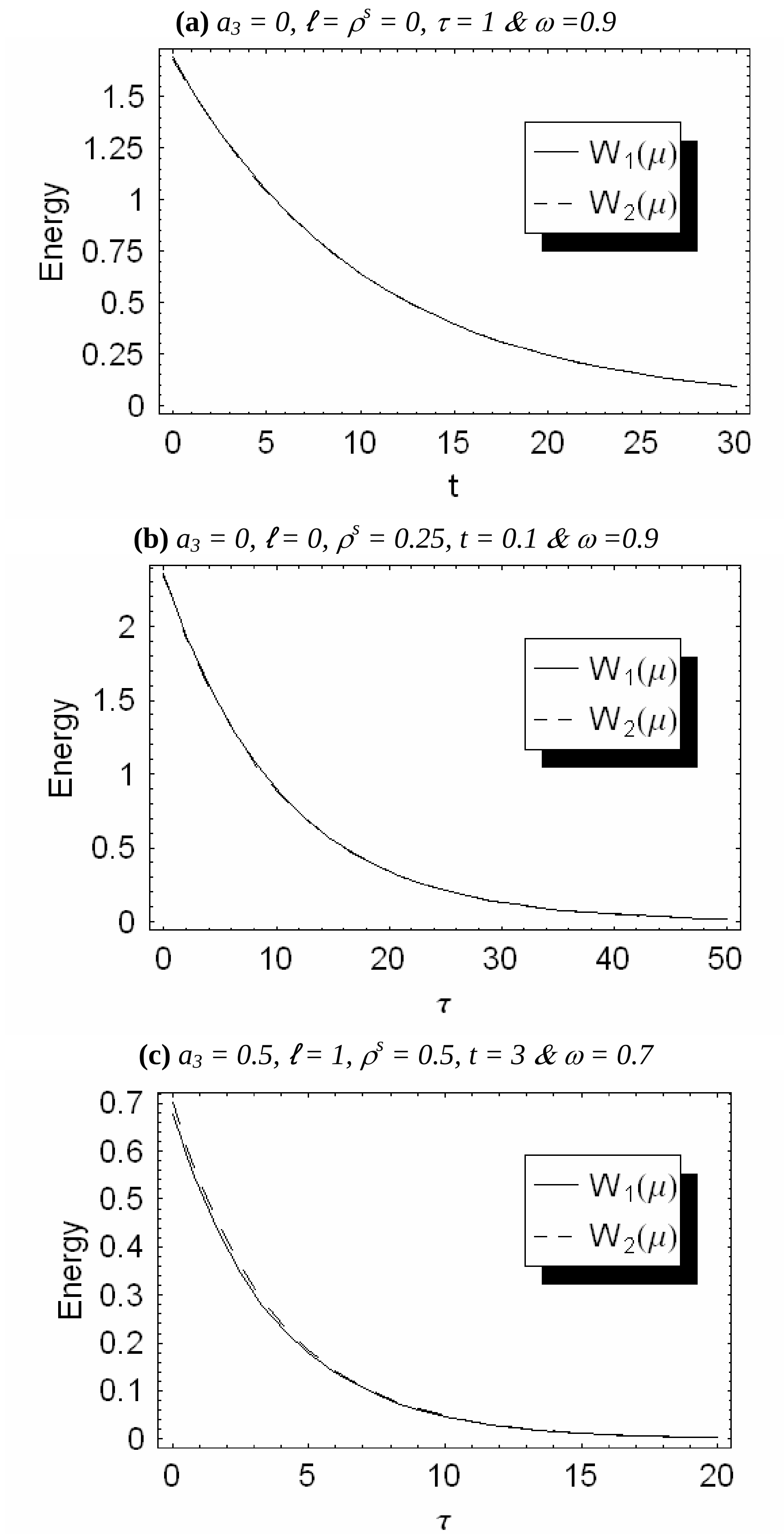
(a) *$a_3 = 0$, $\ell = \rho^s = 0$ & $\omega = 0.9$*



(b) *$a_3 = 0.5$, $\ell = 1$, $\rho^s = 0.5$ & $\omega = 0.7$*

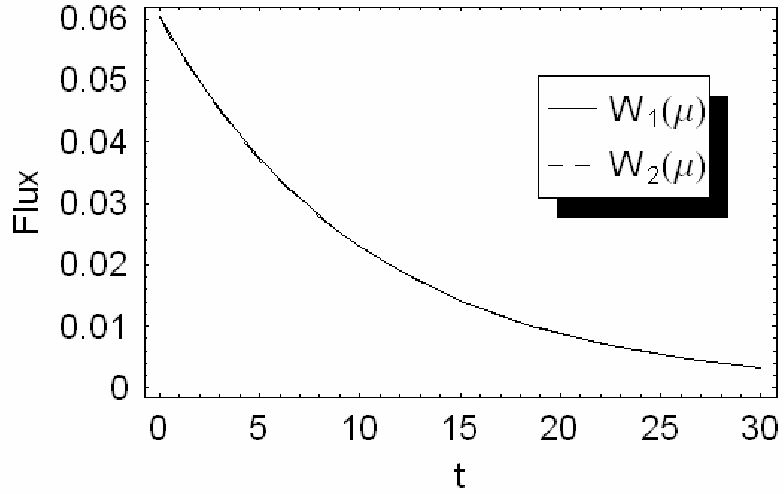


Figs.(2): The reflectivity *R* for pure-triplet scattering

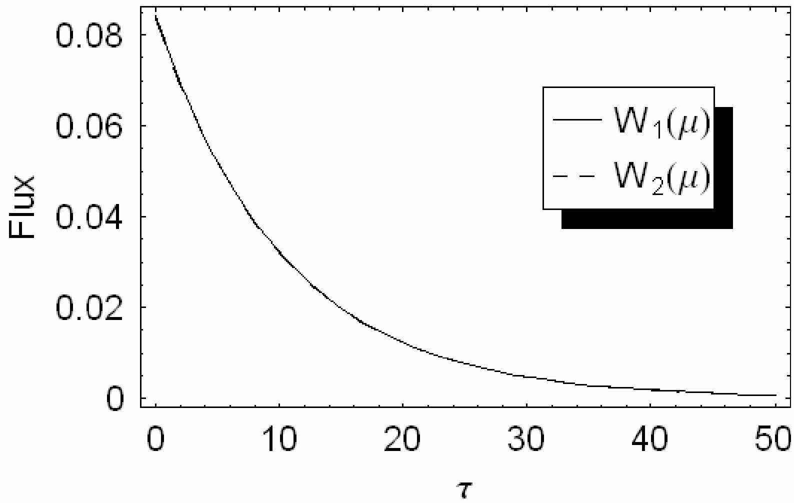


Figs.(3): The neutron radiant energy $E(\tau,t)$ for isotropic and pure-triplet scattering

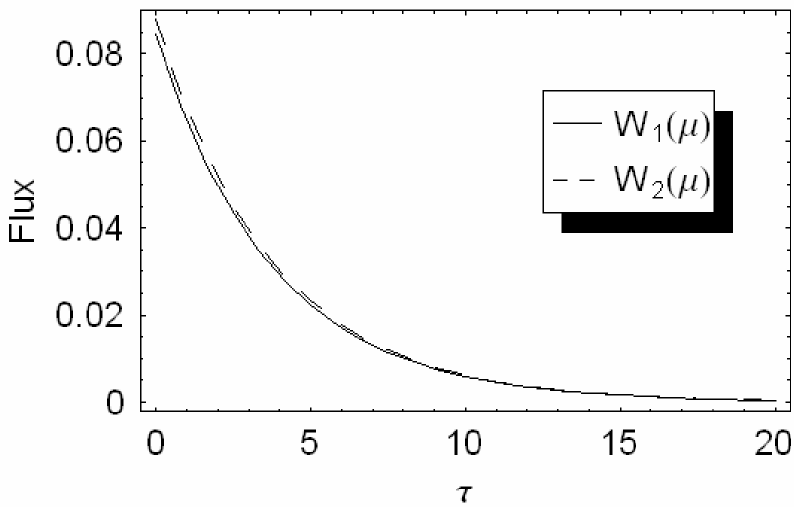
(a) *$a_3 = 0$, $\ell = \rho^s = 0$, $\tau = 1$ & $\omega = 0.9$*



(b) *$a_3 = 0$, $\ell = 0$, $\rho^s = 0.25$, $t = 0.1$ & $\omega = 0.9$*

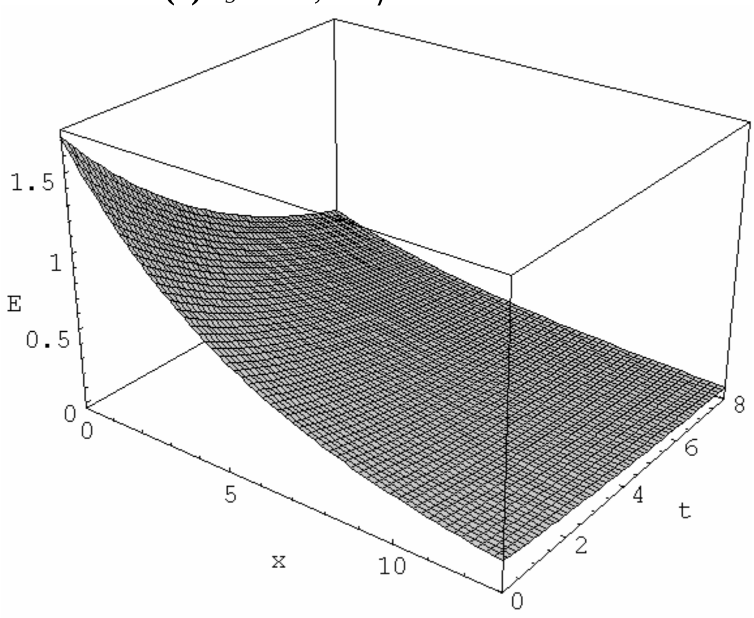


(c) *$a_3 = 0.5$, $\ell = 1$, $\rho^s = 0.5$, $t = 3$ & $\omega = 0.7$*

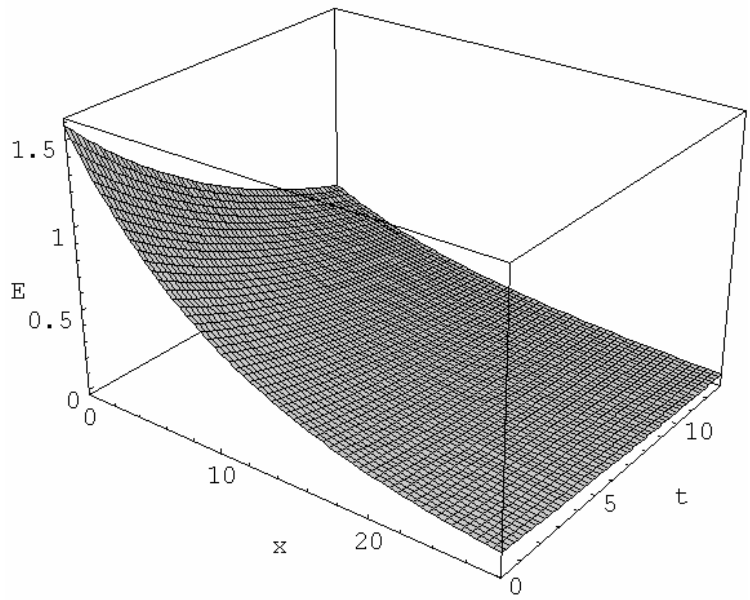


Figs.(4): The neutron net flux *$F(\tau,t)$* for isotropic and pure-triplet scattering

(a) *a_3 = 0.5, $\ell = \rho^s$ = 0 & ω =0.8*

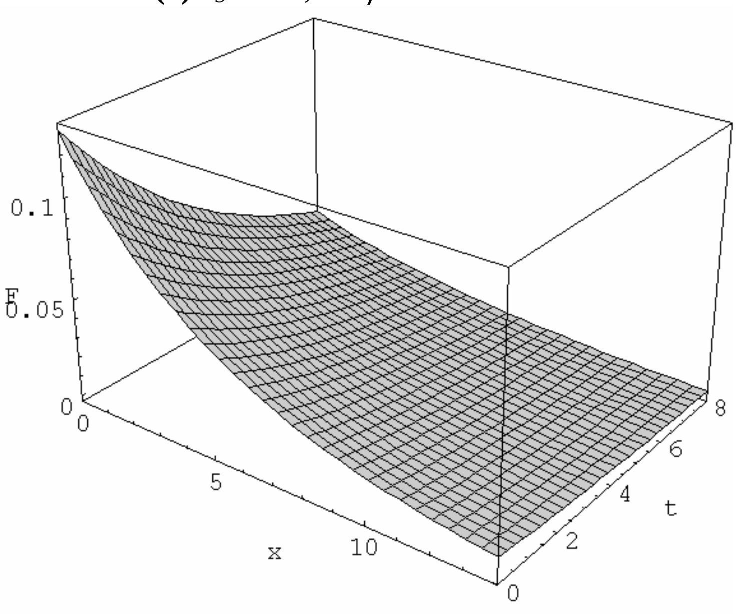


(b) *a_3 = 0, ℓ = 1, ρ^s = 0.25 & ω =0.9*

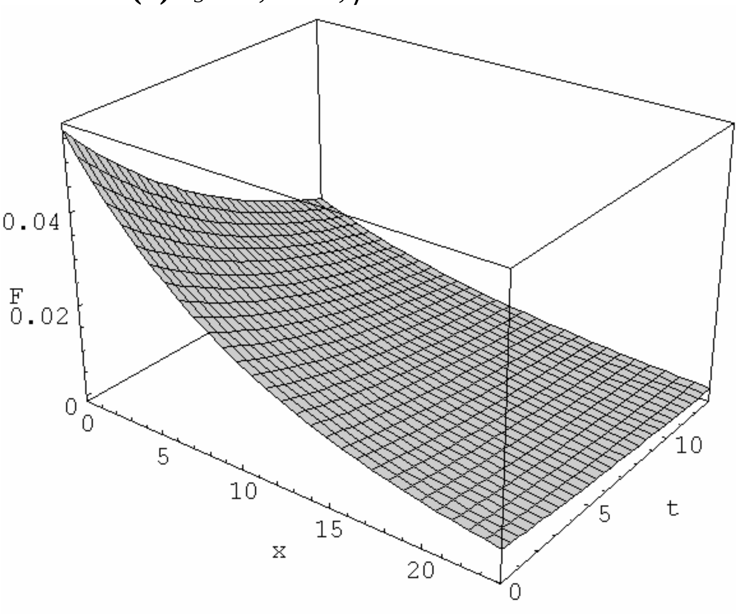


Figs.(5): The neutron radiant energy *$E(\tau,t)$* for isotropic and pure-triplet scattering

(a) $a_3 = 0.5,\ \ell = \rho^s = 0 \ \&\ \omega = 0.8$



(b) $a_3 = 0,\ \ell = 1,\ \rho^s = 0.25 \ \&\ \omega = 0.9$



Figs.(6): The neutron net flux $F(\tau,t)$ for isotropic and pure-triplet scattering

CONCLUSION

The reflectivity, radiant neutron energy and net neutron flux are calculated for time-dependent neutron transport problem in a semi-infinite medium with pure-triplet scattering. The transient equation is transformed into a stationary-like problem and then Case's eigenfunction is obtained for this high order of scattering. The neutron-nucleus scattering provides the pure-triplet scattering. The idea to present a pure-triplet neutron scattering instead of a linear anisotropic scattering is legitimate and motivated in the present work. Two different forms of the weight function are used to force the boundary conditions to be fulfilled. The numerical data give an acceptable results with good agreement between the two weight functions.

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