

# The algebraic derivation and the numerical solution for Faddeev equation for three body quantum systems

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## Abstract

The study of few body quantum system has attracted the attention of the scientific community in order to demonstrat the validity of quantum mechanics in a system more complicated than the h atom ( the atome of helium) several approchoas have been used to treat three body system,the main of them is faddeev equations by their forms complicate and their multiple variable have been proven to be very useful.We shall concentrate on them in this work, we shall present in this work the theoretical derivation of Faddeev equation as well as the numerical solution in order to obtain the waves functions for three body finaly we discuss the result.

# 1 introduction

Three-body problems are treated in various ways. Among them we mention the variational calculation, the hyperspherical formalism and the famous Faddeev equation. In the variational approach one tries to reach the minimum of the expectation value of  $H$  with respect to sufficiently general trial wave functions. In the hyperspherical method the wave function is expanded into a complete set of angular functions on the five-dimensional unit sphere (in the space of the two relative vectors). One is then left with a set of coupled equations in one variable. Finally, the Faddeev equations start with the full knowledge of the two-body sub-systems and work with the two-body  $t$ -matrices. The three-body bound state, an essential physical content of the small three-body problems is the following: the relative motion of two particles is described by the bound state corresponding to their interaction, and consequently the third particle interacts with the two others only by two-body potentials which are averaged over the bound state, the formalism of the small three-body potential problem enters in the form of Hilbert-Schmidt problem, the kernel  $K = (w - T_1 - T_3 - V_1 - V_2)^{-1} V_{12}$ , where  $V_1$  and  $V_2$  are single-particle potentials and  $V_{12}$  a two-body interaction. Eigenvalue problems for Hilbert-Schmidt Kernel in two variables related to the three-body problem are investigated numerically (the treatment of scattering in momentum space requires handling of singularities that reflect the oscillatory behavior of outgoing waves in configuration space while for two-body scattering they are just simple poles, in three-body scattering above the breakup threshold they take the form of so-called moving logarithmic singularities). With the aid of these eigenfunctions the solution of the three-body problem is constructed. In this work we shall discuss in sect 1 the algebraic derivation of the Faddeev equation for three bosons, in sect 2 the numerical solution, finally we discuss

the result.

## 2 the algebraic derivation of Faddeev equation

The ground state of three identical particles which interact via pairwise  $v^i = v_{jk}$  ( $i, j, k = 1, 2, 3$ ) and cyclic permutations thereof is given by Schrodinger equation which reads in integral form

$$|\psi\rangle = G_0 \sum_{i=1}^3 v^i |\psi\rangle \quad (1)$$

$G_0 = (E - H_0)^{-1}$  is the free propagator, where  $H_0$  stands for free hamiltonian and  $E$  for the binding energy of the three-body system.

The decomposition introducing Faddeev composition components

$$|\psi\rangle = \sum_{i=1}^N |\psi_i\rangle \quad (2)$$

Where the Faddeev component is given by:

$$|\psi_i\rangle = G_0 t_i \sum_{j \neq i} |\psi_j\rangle \quad (3)$$

Where  $ijk = 123$  and cyclic permutation thereof, for late clarification we label the coordinates with  $ijk = 123$  as system of type '1', the ones with  $ijk = 231$  as type '2' and the ones with  $ijk = 312$  as type '3'.

The Faddeev components are written in terms of permutation operator as:

$$|\psi_i\rangle = G_0 t_i P |\psi_i\rangle \quad (4)$$

The operator  $t_i$  describes the two-body  $t$  matrix in the subsystem  $jk$  (connect to the pair forces  $v_i$  through the Lipmann-Schwinger equation  $t_i = v_i + v_i G_0 v_i$ ).

The three-nuclear wave function  $|\psi\rangle$  has to be totally symmetric or antisymmetric (case of bosons or fermions respectively) as a consequence, the Faddeev component  $|\psi_i\rangle$  are identical in their functional form.

The permutation operator  $P$  is given as:

$$P = P_{12}P_{23} + P_{13}P_{23} \quad (5)$$

And the total wave function reads:

$$|\psi\rangle = (1 + P)|\psi\rangle \quad (6)$$

In order to solve (4) standard Jacobi momenta is used, the momentum states are normalized and span the space of three-body states which given as:

$$\begin{cases} x_i = x_j - x_k \\ R_i = x_i - \frac{1}{2}(x_j - x_k) \end{cases} \quad (7)$$

The pair  $(r_i, R_i)$  describes completely the relative motions in configuration space and the other two pairs  $(r_k, R_k)$ ,  $i \neq k$  can be expressed linearly in terms of  $(r_i, R_i)$ ,  $k_i$  is the individual momentum of the three particles, then the relative momenta  $(p_i, q_i)$  conjugate to  $(r_i, R_i)$  related to the  $k_i$  given by:

$$\begin{cases} p_i = \frac{1}{\sqrt{2m_j m_k (m_j + m_k)}} (m_k k_j - m_j k_k) \\ q_i = \frac{1}{\sqrt{2m_i M (m_j + m_k)}} (m_i (k_j + k_k) - (m_j + m_k) k_i) \end{cases} \quad (8)$$

Again the pair  $(p_i, q_i)$  describes completely the relative motion in momentum space, the remaining relations follow by cyclic permutation of (8).

The states  $|pq\rangle$  describe the free relative motions of three particles with the help of Jacobi momenta (8). An equivalent description is to use the quantum numbers for the relative orbital angular momentum  $l$  and  $\lambda$  within the pair  $(jk)$  and between

particle  $i$  and the pair  $(jk)$  respectively, together with the magnitudes of  $p$  and  $q$ ,  $L$  and  $S$  are coupled to the total angular momentum  $J$  of the three-nucleon bound state.

For convenience we abbreviate all the discrete quantum numbers by  $\alpha$  and write  $|pq\alpha\rangle$  that basis is complete:

$$\begin{cases} \sum_{\alpha} \int_0^{\infty} dp p^2 \int_0^{\infty} dq q^2 |pq\alpha\rangle \langle pq\alpha| = I \\ \langle pq\alpha | p'q'\alpha' \rangle = \frac{\delta(p-p')}{pp'} \frac{\delta(q-q')}{qq'} \end{cases} \quad (9)$$

Introducing the completeness relations, and explicitly working out the permutation operator  $P$ , we are in position to write down the partial-wave function representation of the Faddeev equation in momentum space. Using two  $\delta$ -functions one finds the Faddeev components:

$$\begin{cases} \psi_{\alpha}(pq) = (E - \frac{p^2}{m} - \frac{3q^2}{4})^{-1} \sum_{\alpha'\alpha''} \int_0^{\infty} q'^2 d\hat{q}' \int_0^{\infty} d\hat{q}'' \\ \int_{-1}^1 dx \frac{\langle pq\alpha | T | \pi_1 q' \alpha' \rangle}{\pi_1'} G_{\alpha'\alpha''}(q'q''x) \frac{\psi_{\alpha''}(\pi_2 q'')}{\pi_2'} \end{cases} \quad (10)$$

with

$$\begin{aligned} \pi_1 &= \sqrt{q'^2 + \frac{q^2}{4} + qq'x} \\ \pi_2 &= \sqrt{q^2 + \frac{q'^2}{4} + qq'x} \end{aligned} \quad (11)$$

And

$$G_{\alpha\alpha'}(qq'x) = \sum_k P_k(x) \sum_{l_1+l_2=l} \sum_{l'_1+l'_2=l'} q^{l_1+l'_1} q^{l_2+l'_2} g_{\alpha\alpha'}^{kl_1l_2l'_1l'_2} \quad (12)$$

the quantity  $g_{\alpha\alpha'}^{kl_1l_2l'_1l'_2}$  is purely geometrical and is given as:

$$\begin{aligned}
g_{\alpha\alpha'}^{kl_1l_2l'_1l'_2} &= \sqrt{\hat{l}\hat{s}\hat{j}\hat{t}\hat{\lambda}\hat{I}\hat{l}'\hat{s}'\hat{j}'\hat{t}'\hat{\lambda}'\hat{I}'(-)} \left\{ \begin{matrix} \frac{1}{2}\frac{1}{2}t \\ \frac{1}{2}Tt' \end{matrix} \right\} \\
&\sum_{LS} \hat{L}\hat{S} \left\{ \begin{matrix} \frac{1}{2}\frac{1}{2}S \\ \frac{1}{2}Ss' \end{matrix} \right\} \left\{ \begin{matrix} lsj \\ \lambda\frac{1}{2}I \\ LSJ \end{matrix} \right\} \left\{ \begin{matrix} l's'j' \\ \lambda'\frac{1}{2}I' \\ L'S'J' \end{matrix} \right\} \hat{k}\left(\frac{1}{2}\right)_2 + l'_1 \\
&\sqrt{\frac{(2l+1)!}{(2l_1)!(2l_2)!}} \sqrt{\frac{(2l'+1)!}{(2l'_1)!(2l'_2)!}} \sum_{ff'} \left\{ \begin{matrix} l_1l_2l \\ \lambda Lf \end{matrix} \right\} c(l_2\lambda f, 00) \\
&\left\{ \begin{matrix} l'_1l'_2l' \\ \lambda' Lf' \end{matrix} \right\} c(l'_1\lambda' f', 00) \left\{ \begin{matrix} fl_1L \\ f'l'_2k \end{matrix} \right\} c(kl_1f', 00)c(kl'_2, 00) \quad (13)
\end{aligned}$$

### 3 the faddeev component of three bosons of 0 spin

Nuclear interaction are short range and therefore act predominantly in states of low orbital angular momentum, the faddeev components for s-wave (ground state) where  $L = 0$   $\lambda = 0$   $l = 0$  are reduce to:

$$\psi(pq) = \frac{1}{E - \frac{p^2}{m} - \frac{3q^2}{m}} \int_0^\infty dq' q'^2 \int_{-1}^1 dx t_0(p, \pi, E - \frac{3q^2}{m}) \psi(\pi_2, q') \quad (14)$$

$E - \frac{3q^2}{4m}$  is the energy available to the interacting two-body subsystem.

The T-three operator is related to the proper two-body t-operator:

$$\langle pq\alpha | T_\alpha | pq\alpha \rangle = \frac{\delta(q-q')}{qq'} \delta_{\alpha\alpha'} t_l(p, p, E - \frac{3q^2}{4m}) \quad (15)$$

wich means the study of three-body system requires the two-body off shell t-matrices:

$$t_l(p, p', z) = v_l(p, p) + \int_0^\infty dp'' p v_l(p, p') \frac{1}{z - \frac{p''^2}{m}} t_l(p, p, z) \quad (16)$$

## 4 Numerical Method

The last integral is easily discretized in the variable  $q$  choice the cut-off  $Q_{max}$  and distributes Gauss-Legendre quadrature point over the interval  $0qQ_{max}$ . Spline interpolation used for numerical values for the  $t$ -matrix which used in order to obtain the numerical solution for Faddeev equation which give the wave function for three bosons.

## References

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