# Noncommutative gravity and application

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## Abstract

From the invariance of the space time noncommutative relations generalized local Lorentz and general coordinates transformations are derived. Moreover, a generalized Klien -Gordon equation is obtained. Applied to the oscillator, it shown that the space-time noncommutativity contributes to a free particle rotator with the moment inertia in rotating in a plane in the effective weak magnetic field perpendicular to the plane of rotation. and the energy eigenvalues are determined exactly.

Keywords: Noncommutative geometry; Klien Gordon oscillator; Gravity

## I. INTRODUCTION

The basic idea of noncommutative geometry is not new and has been known is the context of string theory for some time [1].

We refer the reader to a few of the many excellent reviews of the mathematics of noncommutative space ( [2], [3], [4], [5], [6]).

for a more rigorous understanding of the present material noncommuting coordinates are expected on quite general grounds in any

theory that seeks to in corporate gravity in to a quantum field theory.

Much research has all already in to under understanding noncommutative field theory( [7], [8], [9], [10]) it is equivalent to working with ordinary filed theory and replacing the usual product by the (*) Moylle star product defined as follows.

$$f * g = f \exp\left(\overleftarrow{\partial_{\mu}}\theta^{\mu\nu}\overrightarrow{\partial_{\nu}}\right) g \tag{1}$$

With this definition the space-time coordinates $x_{\mu}$ do $o\left(\theta^{3}\right)$ not commute:

$$[x_{\mu}, x_{\nu}] = i\theta^{\mu\nu} \tag{2}$$

This $*$ product intuitively replaces the point-by-point multiplication of two field by a sort of smeared product and the $\theta$ is an antisymmetric tensor.

this paper is organized as follows : in section 2, the derive the generalized infinitesimal general coordinates and local Lorentz transformations. In section 3, we derive generalized Euler-lagrange field equations and deduce a modified Klein-Gordon equation.

In section 4 the general Klein-Gordon oscillator in a noncommutative, can be represented by ordinary isotropic harmonic oscillator and particle free particle rotator withe the moment of Inertia $\frac{2}{m^{2}\theta^{2}}$ is rotating in a plane in the effective magnetic field weak $\mu = \varpi^{2}\theta$ perpendicular to the plane of rotation.

Finally in section 5, we draw our conclusions.

## II. GENERALIZED INFINITESIMAL GENERAL COORDINATE AND LOCAL LORENTZ TRANSFORMATIONS

Under generalized general coordinate transformations with an infinitesimal noncommutative parameter $\widehat{\xi}$, the coordinates $\widehat{x}^{\mu}$ which has as a representation:

$$\widehat{x}^{\mu} = x^{\mu} + \frac{i}{2}\theta^{\mu\rho}\partial_{\rho} \tag{3}$$

becomes

$$\widehat{x}^{\mu} \rightarrow \widehat{x}^{\prime \mu} = \widehat{x}^{\mu} + \widehat{\xi} \tag{4}$$

with

$$\widehat{\xi}^{\mu} = \xi^{\mu} + \widetilde{\xi}^{\mu} + o(\theta^2) \tag{5}$$

where $\widetilde{\xi}^{\mu}$is an operator which depends on $\theta$ and $\xi^{\mu}$(the ordinary general coordinate transformation parameter). using eqs (1)-(4) and requiring that:

$$\left[x^{\prime \mu}, x^{\prime}_{\nu}\right] = i\theta^{\mu\nu} \tag{6}$$

straightforward simplifications give

$$\left[x^{\prime \mu}, \widetilde{\xi}^{\nu}\right] + \frac{i}{2}\theta^{\mu\rho}\partial_{\rho}\xi^{\nu} = +\frac{i}{2}\theta^{\nu\rho}\partial_{\rho}\xi^{\mu} + [x^{\nu}, \xi^{\mu}] + o(\theta^2, \xi^2) \tag{7}$$

it is to be noted that this last equation , has as a solution

$$\widetilde{\xi}^{\mu} = \frac{i}{2}\theta^{\alpha\rho}\partial_{\alpha}\xi^{\mu}\partial_{\rho} + \frac{i}{2}\theta^{\mu\rho}\partial_{\rho}\xi^{\alpha}\partial_{\alpha} \tag{8}$$

thus, the generalized general coordinate transformations which preserve the canonical noncommutative space-time commutation relation take the form:

$$\widehat{x}^{\prime \mu} = x^{\mu} + \frac{i}{2}\theta^{\alpha\rho}\partial_{\alpha}\xi^{\mu}\partial_{\rho} + \frac{i}{2}\theta^{\mu\rho}\partial_{\rho}\xi^{\alpha}\partial_{\alpha} \tag{9}$$

And under generalized infinitesimal local Lorentz transformations, the noncommutative space-time coordinates $x$ transform as:

$$\widehat{x}^{\mu} \rightarrow \widehat{x}^{\prime \mu} = \widehat{\Lambda}^{\mu}_{L\sigma} x^{\sigma} + \widehat{\Omega}^{\mu} \tag{10}$$

where

$$\widehat{\Lambda}^{\mu}_{L\sigma} = \delta^{\mu}_{\sigma} + \lambda^{\mu}_{\sigma} + \widetilde{\lambda}^{\mu}_{\sigma} \tag{11}$$

and$\widehat{\Omega}^\mu$and.$\widetilde{\lambda}^\mu_\sigma$are operators depending on $\theta$ and $\lambda$.here $\lambda^\mu_\sigma$ denotes the ordinary Lorentz transformation parameter. Now, imposing the preservation of the canonical noncommutative commutation relation (1) , a tedious but direct calculations leads to:

$$\left[\widehat{\Omega}^\mu, x^\nu\right] - \left[\widehat{\Omega}^\nu, x^\mu\right] + i\theta^{\mu\sigma}\lambda^\nu_\sigma - i\theta^{\nu\sigma}\lambda^\mu_\sigma + \left[x^\mu, \widetilde{\lambda}^\nu_\sigma\right] x^\sigma$$

$$\left[-\frac{i}{2}\theta^{\mu\rho}\partial_\rho, \lambda^\nu_\sigma\right] x^\sigma + \left[\frac{i}{2}\theta^{\nu\rho}\partial_\rho, \lambda^\mu_\sigma\right] x^\sigma + o\left(\theta^2, \lambda^2\right) = 0 \tag{12}$$

Its to be noted that this last equation, has as solutions

$$\widehat{\Omega}^\mu = \frac{i}{2}\theta^{\mu\rho}\lambda^\sigma_\rho\partial_\sigma + \frac{i}{2}\theta^{\rho\sigma}\lambda^\mu_\sigma\partial_\rho \tag{13}$$

$$\widetilde{\lambda}^\mu_\sigma = \frac{i}{4}\theta^{\alpha\rho}\partial_\alpha\lambda^\mu_\sigma\partial_\rho + \frac{i}{4}\theta^{\mu\alpha}\partial_\alpha\lambda^\rho_\sigma\partial_\rho \tag{14}$$

Notice that eq. (9) looks like poincaré or an inhomogeneous Lorentz transformation where the translation parameter $x^\mu$is related to the rotation parameter$\lambda^\rho_\sigma$ This means that the generalized local Lorentz transformations in the noncommutative space-time induce at the same time a curvature and a torsion(7).

## III. GENERALIZED FIELD EQUATIONS:

In a general framework of a noncommutative space-time geometry and under infinitesimal gauge variation of any dynamical filed $\hat{\Phi}$, one can write

$$\delta\hat{\Phi} = \widehat{\lambda} * \widehat{G} + \partial_\mu\widehat{\lambda} * T^\mu$$

$$= \delta\Phi + \frac{i}{2}\theta^{\alpha\beta}\left(\partial_\alpha\lambda\partial_\beta G + \partial_\alpha(\partial_\mu\lambda)\partial_\beta T^\mu\right) + \frac{i}{2}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_\gamma\partial_\alpha\lambda\partial_\delta\partial_\beta G + \partial_\gamma\partial_\alpha(\partial_\mu\lambda)\partial_\delta\partial_\beta T^\mu \tag{15}$$

Moreover, the scalar density $\ell$ is a function of the field and their and second and there derivation i.e.

$$\pounds = \pounds\left(\hat{\Phi}, \partial_\mu \hat{\Phi}, \partial_\mu \partial_\nu \hat{\Phi}, \partial_\mu \partial_\nu \partial_\rho \hat{\Phi}\right) \tag{16}$$

and this , the variation of the scalar density under the infinitesimal gauge transformation of $\pounds$ reads

$$\delta\pounds = \delta\hat{\Phi}\frac{\partial\pounds}{\partial\hat{\Phi}} + \delta\left(\partial_\mu\hat{\Phi}\right)\frac{\partial\pounds}{\left(\partial_\mu\hat{\Phi}\right)} + \delta\left(\partial_\mu\partial_\nu\hat{\Phi}\right)\frac{\partial\pounds}{\left(\partial_\mu\partial_\nu\hat{\Phi}\right)} + \delta\left(\partial_\mu\partial_\nu\partial_\rho\hat{\Phi}\right)\frac{\partial\pounds}{\partial\left(\partial_\mu\partial_\nu\partial_\rho\hat{\Phi}\right)} = 0 \tag{17}$$

Now,it is easy to show that the vanishing of the variation of the action lead to the modified equations [11]

$$\frac{.\delta\pounds}{\delta\hat{\Phi}} = \frac{\partial\pounds}{\partial\hat{\Phi}} - \partial_\mu\frac{\partial\pounds}{\left(\partial_\mu\hat{\Phi}\right)} + \partial_\mu\partial_\nu\frac{\partial\pounds}{\left(\partial_\mu\partial_\nu\hat{\Phi}\right)} - \partial_\mu\partial_\nu\partial_\rho\frac{\partial\pounds}{\partial\left(\partial_\mu\partial_\nu\partial_\rho\hat{\Phi}\right)} = 0 \tag{18}$$

## IV. GENERALIZED KLEIN GORDON NONCOMMUTATIVE EQUATION

To determine the Klein Gordon field equation , we consider

$$\pounds = \sqrt{g} * g_{\mu\nu} D_\mu \Phi * D_\nu \Phi + m^2 \Phi * \Phi \tag{19}$$

the Lagrangian scalare field

Where

$$\frac{\partial\ell}{\partial\overline{\Phi}} = \sqrt{g} * \Phi - i\sqrt{g} * g_{\mu\nu} A_\mu D_\nu \Phi + \frac{1}{2}\theta^{\alpha\beta}\sqrt{g} * \partial_\alpha g_{\mu\nu}\partial_\beta\left(A_\mu D_\nu\Phi\right) - \frac{1}{2}\theta^{\alpha\beta}\sqrt{g} * g_{\mu\nu}\partial_\beta A_\mu \partial_\alpha D_\nu \Phi$$

$$+\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}\sqrt{g}(\partial_\gamma\partial_\alpha g_{\mu\nu}\partial_\delta\partial_\beta\left(A_\mu D_\nu\Phi\right) + \partial_\gamma g_{\mu\nu}\partial_\delta(\partial_\beta A_\mu\partial_\alpha D_\nu\Phi))$$

$$+\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}\partial_\gamma\sqrt{g}\partial_\delta(g_{\mu\nu}\partial_\alpha A_\mu\partial_\beta D_\nu\Phi) \tag{20}$$

$$\frac{\partial\ell}{\partial\partial_\mu\overline{\Phi}} = \sqrt{g}*g_{\mu\nu}D_\nu\Phi + \frac{1}{2}\theta^{\alpha\beta}\eta^\alpha_\mu\partial_\beta(\sqrt{g}*\Phi) + \frac{i}{2}\theta^{\alpha\beta}\sqrt{g}*\partial_\alpha g_{\mu\nu}\partial_\beta\left(D_\nu\Phi\right) + \frac{1}{2}\theta^{\alpha\beta}\partial_\alpha\sqrt{g}(g_{\mu\nu}*\partial_\beta D_\nu\Phi)$$

$$+\frac{i}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_{\gamma}\sqrt{g}\partial_{\delta}\partial_{\alpha}g_{\mu\nu}\eta_{\mu}^{\beta}A_{\mu}D_{\nu}\Phi+\sqrt{g}\partial_{\gamma}g_{\mu\nu}\eta_{\mu}^{\delta}\partial_{\alpha}A_{\mu}\partial_{\beta}D_{\nu}\Phi.+\partial_{\gamma}(\sqrt{g}\partial_{\alpha}g_{\mu\nu})\partial_{\beta}\left(A_{\mu}D_{\nu}\Phi\right).\eta_{\mu}^{\delta}$$

$$+\partial_{\delta}A_{\mu}D_{\nu}\Phi\eta_{\mu}^{\beta}+\frac{i}{2}\sqrt{g}\partial_{\gamma}\partial_{\alpha}g_{\mu\nu}\partial_{\delta}\partial_{\beta}\left(D_{\nu}\Phi\right)) \tag{21}$$

$$\frac{\partial\ell}{\partial\partial_{\nu}\overline{\partial_{\mu}\Phi}}=\frac{i}{2}\theta^{\alpha\beta}\sqrt{g}*g_{\mu\nu}\partial_{\beta}D_{\nu}\Phi+\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(2\partial_{\gamma}\sqrt{g}\partial_{\alpha}\Phi\eta_{\mu}^{\beta}\eta_{\nu}^{\delta}+\partial_{\gamma}\partial_{\alpha}\sqrt{g}\Phi\eta_{\mu}^{\beta}\eta_{\nu}^{\delta}+\sqrt{g}\partial_{\delta}\partial_{\alpha}\Phi\eta_{\mu}^{\beta}\eta_{\nu}^{\gamma})$$

$$-\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_{\alpha}\partial_{\gamma}(\sqrt{g}g_{\mu\nu})\partial_{\delta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\beta}+\partial_{\alpha}\partial_{\gamma}(\sqrt{g})\partial_{\delta}\left(g_{\mu\nu}D_{\nu}\Phi\right)\eta_{\nu}^{\beta}+\partial_{\gamma}(\sqrt{g}\partial_{\alpha}g_{\mu\nu})\partial_{\beta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\delta}$$

$$+\partial_{\gamma}(\sqrt{g}\partial_{\alpha}g_{\mu\nu})\partial_{\delta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\beta}+\partial_{\gamma}\sqrt{g}\partial_{\delta}\partial_{\alpha}g_{\mu\nu}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\beta}$$

$$+\partial_{\gamma}(\sqrt{g}g_{\mu\nu})\partial_{\beta}\partial_{\delta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\alpha}-\ \partial_{\alpha}\sqrt{g}\partial_{\gamma}g_{\mu\nu}\left(D_{\nu}\Phi\right)\eta_{\mu}^{\beta}\eta_{\nu}^{\delta})$$

$$+\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}A_{\mu}(\sqrt{g}g_{\mu\nu}\partial_{\beta}\partial_{\delta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\delta}\eta_{\mu}^{\alpha}+2\partial_{\gamma}(\sqrt{g}g_{\mu\nu})\partial_{\beta}\left(D_{\nu}\Phi\right)\eta_{\nu}^{\delta}\eta_{\mu}^{\alpha}+\partial_{\alpha}\partial_{\gamma}(\sqrt{g}g_{\mu\nu})D_{\nu}\Phi\eta_{\nu}^{\delta}\eta_{\beta}^{\alpha} \tag{22}$$

$$\frac{\partial\ell}{\partial\partial_{\rho}\partial_{\nu}\overline{\partial_{\mu}\Phi}}=-\frac{1}{4}\theta^{\alpha\beta}\theta^{\gamma\delta}(\partial_{\alpha}(\partial_{\gamma}\sqrt{g}g_{\mu\nu})D_{\nu}\Phi\eta_{\rho}^{\delta}\eta_{\beta}^{\nu}+\partial_{\gamma}(\sqrt{g}\partial_{\alpha}g_{\mu\nu})D_{\nu}\Phi\eta_{\rho}^{\delta}\eta_{\beta}^{\nu}$$

$$+2\partial_{\gamma}\sqrt{g}g_{\mu\nu}D_{\nu}\Phi\eta_{\rho}^{\delta}\eta_{\alpha}^{\nu}+\sqrt{g}g_{\mu\nu}\partial_{\beta}\partial_{\delta}D_{\nu}\Phi\eta_{\rho}^{\gamma}\eta_{\alpha}^{\nu} \tag{23}$$

In a noncommutative Minkowski space in which the metric specified by$dig\left(-+++\right)$ one may describe the Klein Gordon Oscillator using the following equation (20, 21, 22, 23)in shut that $A\left(0,mwx\right)$ we obtain :

$$\left\{\left[1+\frac{m^{2}\varpi^{2}}{4}\left(\left(\theta^{12}\right)^{2}+\left(\theta^{13}\right)^{2}\right)\right]p_{1}^{2}+\left[1+\frac{m^{2}\varpi^{2}}{4}\left(\left(\theta^{12}\right)^{2}+\left(\theta^{23}\right)^{2}\right)\right]p_{2}^{2}\right.$$

$$+\left[1+\frac{m^2\varpi^2}{4}\left(\left(\theta^{32}\right)^2+\left(\theta^{13}\right)^2\right)\right]p_3^2$$

$$+m^2\varpi^2\left(x^2+y^2\right)+im^2\varpi^2\epsilon_{ijk}\theta^{ij}L_k\Big\}\Phi=\left(E^2-m^2+3m\varpi\right)\Phi \tag{24}$$

If we put $\theta^{12}=\theta$ and choose the rest of the $\theta$ components equal to zero ( which can be done by a rotating in the Z direction) we arrive at the following equation which can be solved exactly:

$$\left\{\left[1+\frac{m^2\varpi^2}{4}\left(\theta^2\right)\right]\left(p_1^2+p_2^2\right)+p_3^2+m^2\varpi^2\left(x^2+y^2\right)-m^2\varpi^2\theta L_3\right\}\Phi=\left(E^2-m^2+3m\varpi\right)\Phi \tag{25}$$

The above equation can be solved exactly and has a similar behavior to the dynamics of a free particle rotator with the moment inertia $\frac{2}{\varpi^2\theta^2}$ in rotating in a pane in the effective weak magnetic field $\mu=\varpi^2\theta^2$ perpendicular to the plane of rotation. In should be noted a noncommutative space is not isotropic and the energy eigenvalues are given by:

$$E^2=2m\varpi\left(n+3/2\right)+m^2-3m\varpi+\frac{m\varpi\theta}{2}\left(n_1+n_2+1\right)-m^2\varpi^2\theta m_l \tag{26}$$

## V. CONCLUSION

Throughout out this work we have first constructed generalized infinitesimal local coordinates and Lorentz transformation which preserve the noncommutative coordinates canonical commutation relation . It turns out that the form of the latter, looks like classical inhomogeneous Lorentz transformation. This suggests that noncommutativity induces gravity .Second we have generalized the equations of motion and derive a generalized field equations. we have shown the appears of the new term in Hamiltonian of the oscillator in a noncommutative space [12] which can be interpreted as a free particle rotator with the moment inertia $\frac{2}{\varpi^2\theta^2}$ in rotating in a pane in the effective weak magnetic field $\mu=\varpi^2\theta^2$ perpendicular to the plane of rotation.

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