Journal of Nuclear and Radiation Physics, Vol. 2, No. 2, 2007, pp. 89-102

GENERALIZED ANALYTICAL TREATMENT OF THE SOURCE STRENGTH IN THE SOLUTION OF THE DIFFUSION EQUATION

Khaled S. M. Essa, and Maha S. El-Otaify

Department of Mathematics and Theoretical Physics, Nuclear Research Center, Atomic Energy Authority, P.O. 13759, Cairo-Egypt

Rec. 19/6/2007 *Accept. 3/9/2007*

The source release strength (which is an integral part of the mathematical formulation of the diffusion equation) together with the boundary conditions leads to three different forms of the diffusion equation. The obtained forms have been solved analytically under different boundary conditions, by using transformation of axis, cosine, and Fourier transformation. Three equivalent alternative mathematical formulations of the problem have been obtained. The estimated solution of the concentrations at the ground source has been used for comparison with observed concentrations data for SF_6 tracer experiments in low wind and unstable conditions at IIT Delhi sports ground. A good agreement between estimated and observed concentrations is found.

Keywords: *Eddy Diffusivity / Fourier Transformation / Boundary conditions of diffusion Equation*

INTRODUCTION

The Gaussian plume models have been used extensively for air quality predictions especially by the regulatory authorities all over the world. The popularity of these models for estimating air pollutant concentrations from different sources stems from their simplicity, easy to understand and use and computational efficiency. Extensive information on these models can be found in the literature [1-4]. The forms of the Gaussian plume solution and the mathematical problem associated with it have been widely discussed in [2,5].

Recently, it has been emphasized [6-10], that for the treatment of near source dispersion, eddy diffusivities are dependent on a linear function of down wind distance from the source. This has been used in dispersion models in convective [7], and stable weak wind conditions [8].

An analytical approach for the solution of the advection-diffusion equation using the Laplace transform technique with analytical inversion is solved by Wortmann[11].

An important aspect of the problem is the representation of the source in the model formulation. The source term can be accounted, in two possible ways: either through the material balance or through the boundary of the domain. The former seems quite obvious physically whereas the latter needs elaborate description mathematically. Sometimes, the source term is accounted for through the material balance equation even though it lies on the boundary [12] of the domain. The other way commonly found in the literature to account for the source is through one of the boundaries (x=0 or z=0 in the case of ground level release) based purely on physical considerations.

It is relatively easier to obtain a closed form solution of the resulting problem by accounting the source using one of the possible ways.

In this work, we solved three different forms of the diffusion-advection equation under different boundary conditions using transform axis**,** cosine, and Fourier transform. After we had solved the problems, we got about three equivalent alternative mathematical formulations. The obtained solution for the ground source has been used to compare with observed concentrations data for SF_6 tracer experiments in low wind and unstable conditions at IIT Delhi sports ground. Good agreements between estimated and observed concentrations are obtained.

STUDY THE BOUNDARY CONDITIONS OF DIFFUSION EQUATION

We are going to study the boundary conditions of the diffusion equation in the three- dimensional [12] and finding the equivalent problems of it, with different boundary conditions together the source term.

1.1-First Case (General State)

The diffusion equation in the steady-state, and when the x-axis is oriented in the direction of the mean wind; can be written as:

$$U\frac{\partial C(x,y,z)}{\partial x}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)+Q\delta(x)\delta(y)\delta(z) \qquad (1)$$

where $\delta(\ldots)$ is Dirac's delta function, $C\equiv C(x,y,z)$ is the mean pollutant concentration, U is the mean wind speed directed towards the positive x-axis and thus we have assumed $U > 0$, Q is the source strength, and K_x, K_y and K_z are the eddy diffusivities in the x-, y- and z-directions, respectively, generally, in the direction of mean wind.

Consider the following boundary conditions:

(i) $C \to 0$ as $|x|,\ |y|,\ z \to \infty$ (2a)

(ii) C = 0 at x=0 (deleted neighborhood) (2b)

at z=0 (deleted neighborhood) $K_z \dfrac{\partial C(x,y,z)}{\partial z} = 0$ (iii) - (2c)

where, deleted neighborhood means the region excluding a small neighborhood of the point where the source is located.

Integrating Eq.(1) with respect to y from $-\infty \to \infty$, using the boundary condition (2a) , then integrating with respect to z from $0 \to \infty$, using the boundary conditions (2a) and (2c), and finally, integrating with respect to x from $-\infty \to x$; one gets:

$$\int_0^\infty \int_{-\infty}^\infty \left[UC(x,y,z) - K_x \frac{\partial C}{\partial x} \right] dy\, dz = Q \qquad \forall\, x \geq 0 \tag{3a}$$

$$= 0 \qquad \forall\, x < 0 \tag{3b}$$

This means that, for all negative values of x, UC = 0, which implies C = 0 $\forall\, x<0$, and $\forall\, x>0$ the net flux in x-direction across any plane normal to x-axis, equals the source strength Q, and so this term is no longer contribute in Eq.(1), and we can rewrite Eq.(3a) at x=0 according to the continuity of the flux Q in the equivalent form:

$$UC(x,y,z) - K_x \frac{\partial C}{\partial x} = Q\delta(y)\delta(z) \tag{4}$$

Therefrom, Eq.(1) with its boundary conditions(2a-c) can be written in the following equivalent case.

1.2-Second Case

The diffusion equation is writing, as:

$$U \frac{\partial C(x,y,z)}{\partial x} = \frac{\partial}{\partial x}\left(K_x \frac{\partial C}{\partial x} \right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial C}{\partial z} \right) \tag{5}$$

with the following boundary conditions:

(i) $C \to 0$ as $|x|,\ |y|,\ z \to \infty$ (6a)

(ii) $UC(x,y,z) - K_x \dfrac{\partial C}{\partial x} = Q\delta(y)\delta(z)$ at x = 0 (6b)

at z = 0 (deleted neighborhood) $K_z \dfrac{\partial C(x,y,z)}{\partial z} = 0$ (iii) - (6c)

which is different of that recently proved [12], by the second term in the left hand side of Eq 6b.

Now, we are going to find another equivalent form of the diffusion equation (1). This can be accomplished by integrating Eq.(1) with respect to y from $-\infty \to \infty$, using the boundary condition (2a) , then integrating with respect to x from $0 \to \infty$, using the boundary conditions (2b) and (2a), and integrating with respect to z from $0 \to z$, ; one gets:

$$z > 0 \ \forall = Q \qquad \int_0^\infty \int_{-\infty}^\infty \left(-K_z \frac{\partial C(x,y,z)}{\partial z} \right) dy\, dx \tag{7}$$

For z > 0 the, the double integral in Eq.(7)represents the flux across any horizontal plane. In this region, the source term will no longer contribute in Eq.(1). Eq.(1), then can be solved with an appropriate boundary condition at z = 0. Because of the continuity of the flux Q of pollutant for z > 0, Eq.(7) must hold even at z = 0. Thus, the boundary condition at z = 0 may be written as:

$$-K_z \frac{\partial C(x,y,z)}{\partial z} = Q\delta(x)\delta(y), \tag{8}$$

and we have the following equivalent case.

1.3-Third Case

The diffusion Eq.(5) with the following boundary conditions:

(i) $C \to 0$ as $|x|$, $|y|$, $z \to \infty$ (9a)

(ii) $C = 0$ at $x = 0$ (deleted neighborhood) (9b)

(iii) $-K_z \frac{\partial C(x,y,z)}{\partial z} = Q\delta(x)\delta(y)$ at $z = 0$ (9c)

represents the third case.

DIFFUSION EQUATION SOLUTIONS UNDER CONSTANT EDDY DIFFUSIVITIES

We are going to solve the three above different cases of the diffusion equations.

2.1- Solution of the First Case:

Consider Eq.(1) with its boundary conditions as given by Eq.(2a-c), together with constants K_x, K_y, and K_z. To solve this equation, let us transform the coordinates x, y, and z to the coordinates X,Y, and Z as:

$$X = x/\sqrt{K_x} \quad , \quad Y = y/\sqrt{K_y} \quad , \quad Z = z/\sqrt{K_z} \tag{10}$$

Eq.(1) under the above transformations, and after using the relation[13] [$\delta(ax)=\delta(x)/|a|$] becomes:

$$U_* \frac{\partial C(X,Y,Z)}{\partial X} = \frac{\partial^2 C}{\partial X^2} + \frac{\partial^2 C}{\partial Y^2} + \frac{\partial^2 C}{\partial Z^2} + Q^* \delta(X)\delta(Y)\delta(Z) \tag{11}$$

where

$$U_* = U/\sqrt{K_x} \quad , \quad Q^* = Q/\sqrt{K_x K_y K_z} \tag{12}$$

Upon, using cosine transformation in Z [14], and the boundary conditions (2b) and (2c), Eq.(11) becomes:

$$U_* \frac{\partial \bar{C}(X,Y,\lambda_3)}{\partial X} = \frac{\partial^2 \bar{C}}{\partial X^2} + \frac{\partial^2 \bar{C}}{\partial Y^2} - \lambda_3^2 \bar{C} + \sqrt{\frac{2}{\pi}} Q^* \delta(X)\delta(Y) \tag{13}$$

in which $\bar{C}(X,Y,\lambda_3)$ is the cosine transformed function in λ_3; namely:

$$\bar{C}(X,Y,\lambda_3) = \sqrt{2/\pi} \int_0^\infty C(X,Y,Z)\cos\lambda_3 Z \, dZ \tag{14}$$

Now, using Fourier and cosine transformation in X and in Y respectively in Eq.(13), and the boundary condition (2b) ; to get:

$$U_*[-i\lambda_1 \tilde{C}(\lambda_1 \lambda_{2,} \lambda_3)] = -\lambda_1^2 \tilde{C} - \lambda_2^2 \tilde{C} - \lambda_3^2 \tilde{C} + Q^* \sqrt{2/\pi^3}$$

which after solving for, $\tilde{C}(\lambda_1,\lambda_2,\lambda_3)$, yields:

$$\tilde{C}(\lambda_1 \lambda_{2,} \lambda_3) = \sqrt{\frac{2}{\pi}} \frac{Q^*}{\pi\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - i\lambda_1 U_*\right]} \tag{15}$$

in which $\tilde{C}(\lambda_1,\lambda_2,\lambda_3)$ is Fourier and cosine transformed function in λ_1 and λ_2 , namely:

$$\tilde{C}(\lambda_1 \lambda_{2,} \lambda_3) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \bar{C}(X,Y,\lambda_3) e^{i(\lambda_1 X)} \cos\lambda_2 Y \, dX \, dY \tag{16}$$

Now, using the inverse Fourier and cosine transformation in λ_1 and λ_2 , and employing Eq.(15), to get:

$$\bar{C}(X,Y,\lambda_3) = \sqrt{\frac{2}{\pi}} \frac{Q^*}{\pi^2} \int_0^\infty \int_{-\infty}^\infty \frac{e^{-i(\lambda_1 X)} \cos\lambda_2 Y}{\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - i\lambda_1 U_*\right]} d\lambda_1 \, d\lambda_2 \tag{17}$$

First employing the integrations on λ_2 , then after transforming λ_1 to [λ_1 - ($iU_*/2$)], and performing the second integration on λ_1 in terms of cosine [15,16], we obtain:

$$\bar{C}(X,Y,\lambda_3)=\sqrt{\frac{2}{\pi}}\frac{Q^*}{\pi}\exp\left[\frac{U_*X}{2}\right]K_0\left[\left(X^2+Y^2\right)^{\frac{1}{2}}\left(\lambda_3^2+\frac{U_*^2}{4}\right)^{\frac{1}{2}}\right] \tag{18}$$

where K_0 is the modified Bessel function of the second kind of zero order.

Now, we are in the position to find C(X,Y,Z). This can be accomplished by inverting Eq.(14) with respect to λ_3, and using Eq.(18), to get:

$$C(X,Y,Z)=\frac{2Q^*}{\pi^2}\exp\left[\frac{U_*X}{2}\right]\int_0^\infty K_0\left[\left(X^2+Y^2\right)^{\frac{1}{2}}\left(\lambda_3^2+\frac{U_*^2}{4}\right)^{\frac{1}{2}}\right]\cos\lambda_3 Z\,d\lambda_3 \tag{19}$$

The integration in Eq.(19), can be evaluated from [16]; consequently C(X,Y,Z) can be explicit explicitly as:

$$C(X,Y,Z)=\frac{Q^*}{\pi}\exp\left[\frac{U_*X}{2}\right]\left(X^2+Y^2+Z^2\right)^{\frac{-1}{2}}\exp\left[\frac{-U_*\sqrt{X^2+Y^2+Z^2}}{2}\right] \tag{20}$$

and after express of the values of X,Y,Z as given in Eq.(10), and the values of U_* and Q^* as given in Eq.(12), C(x,y,z), reads:

$$C(x,y,z)=\frac{Q}{\pi\sqrt{K_xK_yK_z}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}}\exp\left[\frac{xU}{2K_x}\right]\exp\left[\frac{-U}{2\sqrt{K_x}}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}\right] \tag{21}$$

which is the general solution of the diffusion Eq.(1).

2.2 - Solution of the Second Case:

We rewrite the diffusion equation as given in Eq.(5), as:

$$U\frac{\partial C_*(x,y,z)}{\partial x}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C_*}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C_*}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C_*}{\partial z}\right) \tag{22}$$

in which K_x, K_y and K_z are constant, and with the following boundary conditions:

(i) $C_*\to 0$ as $|x|$, $|y|$, $z\to\infty$ (23a)

(ii) $UC_*(x,y,z)-K_x\frac{\partial C_*}{\partial x}=Q\delta(y)\delta(z)$ at x = 0 (23b)

(iii) $-K_z\frac{\partial C_*(x,y,z)}{\partial z}=0$ at z = 0 (deleted neighborhood) (23c)

Eq.(22) can be solved by using Fourier transformation in y, and cosine transformation in z to give:

$$U\frac{\partial \overline{C}_*(x,\overline{\lambda}_2,\overline{\lambda}_3)}{\partial x} = K_x\frac{\partial^2 \overline{C}_*}{\partial x^2} - K_y\overline{\lambda}_2^2\overline{C}_* - K_z\overline{\lambda}_3^2\overline{C}_* \tag{24}$$

in which $\overline{C}_*(x,\overline{\lambda}_2,\overline{\lambda}_3)$ is the Fourier and cosine transformed function in $\overline{\lambda}_2$ and $\overline{\lambda}_3$, namely:

$$\overline{C}_*(x,\overline{\lambda}_2,\overline{\lambda}_3) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} C_*(x,y,z)e^{i\overline{\lambda}_2 y}\cos\overline{\lambda}_3 z\, dy\, dz \tag{25}$$

Eq.(24) represents a partial differential equation of second order in x, with constant coefficients and has a general solution in the form:

$$\overline{C}_*(x) = A\, e^{\left[K_*^+ x\right]} + B\, e^{\left[K_*^- x\right]} \tag{26}$$

where A and B are constants, and $K_*^{\pm}$, stands for:

$$K_*^{\pm} = \frac{U \pm \sqrt{U^2 + 4K_x\left(K_y\overline{\lambda}_2^2 + K_z\overline{\lambda}_3^2\right)}}{2K_x} \tag{27}$$

The constants, A and B can be determined from the boundary conditions as given by Eqs.(23a-c). Upon, introducing the boundary condition (23b), and (23a), in Eq.(25) respectively, we have the boundary conditions of $\overline{C}_*(x)$, namely:

$$\left(U\overline{C}_*(x) - K_x\frac{\partial \overline{C}_*(x)}{\partial x}\right)\Bigg|_{x=0} = \frac{Q}{\pi} \qquad \text{and} \qquad \overline{C}_*(x) = 0 \quad \text{as} \quad x \to \infty \tag{28}$$

Introducing Eq.(27) in Eq.(26), then after putting at $x \to \infty$ and x=0 respectively, and employing Eq.(28); we have:

$$A=0 \qquad \& \qquad B = \frac{2Q}{\pi\left[U + \sqrt{U^2 + 4K_x\left(K_y\overline{\lambda}_2^2 + K_z\overline{\lambda}_3^2\right)}\right]} \tag{29}$$

Introducing the above equation in Eq.(26); to yields:

$$\overline{C}_*(x,\overline{\lambda}_2,\overline{\lambda}_3) = \exp\left[\frac{xU}{2K_x}\right]\frac{Q\exp\left[-\frac{x}{\sqrt{K_x}}\sqrt{\left(K_z\overline{\lambda}_3^2 + \left[U^2/4K_x\right]\right) + \left(K_y\overline{\lambda}_2^2\right)}\right]}{\pi\sqrt{K_x}\left[\left(U/2\sqrt{K_x}\right) + \sqrt{\left(K_z\overline{\lambda}_3^2 + \left[U^2/4K_x\right]\right) + \left(K_y\overline{\lambda}_2^2\right)}\right]} \tag{30}$$

Inverting Eq.(25) with respect to $\overline{\lambda}_2$ and $\overline{\lambda}_3$, and using Eq.(30), to get:

$$C_*(x,y,z) = \frac{Q}{\pi^2\sqrt{K_x}}\exp\left[\frac{xU}{2K_x}\right] \tag{31}$$

$$\int_0^\infty \int_{-\infty}^\infty \frac{\exp\left[-\frac{x}{\sqrt{K_x}}\sqrt{\left(K_z\bar{\lambda}_3^2+\left[U^2/4K_x\right]\right)+\left(K_y\bar{\lambda}_2^2\right)}\right]}{\left[\left(U/2\sqrt{K_x}\right)+\sqrt{\left(K_z\bar{\lambda}_3^2+\left[U^2/4K_x\right]\right)+\left(K_y\bar{\lambda}_2^2\right)}\right]} e^{-i\bar{\lambda}_2 y}\cos\bar{\lambda}_3 z\, d\bar{\lambda}_2 d\bar{\lambda}_3$$

The integration on $\bar{\lambda}_2$ can be evaluated after replacing the exponentional form with the cosine form, replacing $\bar{\lambda}_2$ with $\left(\lambda/\sqrt{k_y}\right)$, and using [16] to give:

$$C_*(x,y,z)=\frac{2Q}{\pi^2\sqrt{K_x}\sqrt{K_y}}\exp\left[\frac{xU}{2K_x}\right]\int_0^\infty K_0\left[\left(\frac{x^2}{K_x}+\frac{y^2}{K_y}\right)^{\frac{1}{2}}\left(K_z\bar{\lambda}_3^2+\frac{U^2}{4K_x}\right)^{\frac{1}{2}}\right]\cos\bar{\lambda}_3 z\, d\bar{\lambda}_3 \qquad (32)$$

The integration in the above equation is similar to that given in Eq.(19) ; and consequently by of its value in Eq.(20), the above equation reads:

$$C_*(x,y,z)=\frac{Q}{\pi\sqrt{K_xK_yK_z}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}}\exp\left[\frac{xU}{2K_x}\right]\exp\left[\frac{-U}{2\sqrt{K_x}}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}\right] \qquad (33)$$

which is equivalent to what has been obtained previously in Eq.(21).

2.2 - Solution of the Third Case:

Rewriting the diffusion equation as given in Eq.(5), as:

$$U\frac{\partial C_{**}(x,y,z)}{\partial x}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C_{**}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C_{**}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C_{**}}{\partial z}\right) \qquad (34)$$

in which $K_{x,}$ $K_{y,}$ and K_z are constant, and with the following boundary conditions:

(i) $C_{**}\to 0$ as $|x|$, $|y|$, $z \to\infty$ (35a)

(ii) $C_{**} = 0$ at $x = 0$ (deleted neighborhood) (35b)

(iii) $-K_z\frac{\partial C_{**}(x,y,z)}{\partial z} = Q\delta(x)\delta(y)$ at $z = 0$ (35c)

Using Fourier transformation in x and cosine transformation in y in Eq.(34), then after using the boundary conditions (35a) ; Eq.(34), becomes:

$$-i\tilde{\lambda}_1 U\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)=-\tilde{\lambda}_1^2 K_x\bar{C}_{**}-\tilde{\lambda}_2^2 K_y\bar{C}_{**}+K_z\frac{\partial^2\bar{C}_{**}}{\partial z^2} \qquad (36)$$

in which $\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)$ is the Fourier and cosine transformed function in $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, namely:

(37)

$$\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)=(1/\pi)\int_0^{\infty}\int_{-\infty}^{\infty}C_{**}(x,y,z)e^{i\tilde{\lambda}_1 x}\cos\tilde{\lambda}_2 y\,dx\,dy$$

K_* has the form:

$$K_*^2=\frac{i\tilde{\lambda}_1 U-\tilde{\lambda}_1^2 K_x-\tilde{\lambda}_2^2 K_y}{K_z} \tag{38}$$

Eq.(36) represent a partial differential equation of second order in z, with constant coefficients and absence of the first derivative. It has a general solution on the form:

$$\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)=A\,e^{iK_* z}+B\,e^{-iK_* z} \tag{39}$$

where A and B are constants, can be determined from the boundary conditions as given by Eqs.(39a-c). Introducing the boundary condition (35a), and (35c), in Eq.(37), determine the boundary conditions of $\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)$, namely:

$$\bar{C}_{**}\to 0 \quad \text{as} \quad z\to\infty \tag{40}$$

and

$$\left(-K_z\frac{\partial\bar{C}_{**}(z)}{\partial z}\right)\Bigg|_{z=0}=\frac{Q}{\pi} \qquad \text{at} \quad z=0 \tag{41}$$

Upon putting $z\to\infty$ in Eq.(39) and z=0 in its derivative ; one gets:

$$A=0 \qquad \& \qquad \left.\frac{\partial\bar{C}_{**}(z)}{\partial z}\right|_{z=0}=i\,K_*(A\text{-}B) \tag{42}$$

Introducing Eq.(41) together Eq.(42) in Eq.(39) ; to reads:

$$\bar{C}_{**}(\tilde{\lambda}_1,\tilde{\lambda}_2,z)=\left(Q/i\pi K_z K_*\right)e^{-iK_* z} \tag{43}$$

Inverting Eq.(37) with respect to $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$, and using Eq.(43), to get:

$$C_{**}(x,y,z)=\int_0^{\infty}\int_{-\infty}^{\infty}\left(Q/i\pi^2 K_z K_*\right)e^{-iK_* z}e^{-i\tilde{\lambda}_1 x}\cos\tilde{\lambda}_2 y\,d\tilde{\lambda}_1\,d\tilde{\lambda}_2 \tag{44}$$

Let us evaluate the integration on $\tilde{\lambda}_1$ by transforming first $\tilde{\lambda}_1$ to λ_* by putting [-i $\tilde{\lambda}_1\sqrt{K_x}=\lambda_*$], then transform λ_* to $\tilde{\lambda}$ by putting [$\tilde{\lambda}=\sqrt{\lambda_*^2-\left(\lambda_* U/\sqrt{K_x}\right)-\lambda_2^2 K_y}$], in which the integration on λ_* is similar to that given in Eq.(31) ; and consequently by of analogy with Eq.(32), one gets:

$$\int_{-\infty}^{\infty}\frac{e^{-iK_* z}e^{-i\lambda_1 x}}{K_*}d\lambda_1=\frac{i\sqrt{K_z}\exp\left(xU/2\sqrt{K_x}\right)}{\sqrt{K_x}}\quad K_0\left[\left(\frac{x^2}{K_x}+\frac{z^2}{K_z}\right)^{\frac{1}{2}}\left(K_y\lambda_2^2+\frac{U^2}{4K_x}\right)^{\frac{1}{2}}\right] \tag{45}$$

Introduction Eq.(45) in Eq.(44), by virtue of Eq.(36), yield:

$$C_{**}(x,y,z) = \frac{Q}{2\pi\sqrt{K_x K_y K_z}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}} \exp\left[\frac{xU}{2K_x}\right]\exp\left[\frac{-U}{2\sqrt{K_x}}\sqrt{\frac{x^2}{K_x}+\frac{y^2}{K_y}+\frac{z^2}{K_z}}\right] \quad (46)$$

which is equivalent to what has been obtained previously in Eq.(21)and Eq.(33).

Therefore, in this section we have proved that the solution of the three previous different diffusion equations, are equivalent.

3-Case Study

By assuming the K's as linear functions of downwind distance $K_x = \alpha Ux$, $K_y = \beta Ux$ $K_z = \gamma Yx$), where α, β and γ are constants which is estimated from [9] as follows:

$$\alpha = \beta = 0.3(w_* / U)^2; \qquad \gamma = 0.16(w_* / U)^2 \quad (47)$$

where w_* is the convective velocity scale. The relations (47), for the convective conditions, have been used in solution (33) for estimating diffusion experiments conducted at IIT Delhi for ground-level releases during low wind conditions [17].

The air samples thus collected were later analyzed in the Air Pollution Lab (Dry), CAS, IIT Delhi, using electron –capture gas chromatography [17]. Meteorological inputs have been provided by the measurements done at 1, 2, 4, 8, 15, and 30m levels at a 30m micrometeorological tower located about 300m south-east of the release point. Table 1. gives the relevant information about the diffusion tests and the wind vectors. In addition, it includes values of w_*. The data from these 8 unstable test runs have been utilized for the following analysis.

Table 1. Relevant experimental details of the convective test runs conducted at IIT Delhi sports ground in February 1991

Run no.	Sampling Time(h)	Wind Speed (m/s)	Wind direction (deg)	w_* (m/s)	P-G stability class
1	1200-1230	1.36	343	2.37	A-B
2	1530-1600	0.74	291	2.26	B
6	1000-1030	1.40	286	2.04	B
7	1245-1315	1.54	284	2.28	B
8	1645-1715	0.89	301	1.09	B
11	1000-1030	1.07	230	1.83	A-B
12	1215-1245	1.55	334	2.32	B
13	1530-1600	1.08	331	1.72	B

Tables 2. and 3. include the results from slender plume approximation (α→0) which is the same as the Gaussian plume formula based on the above mentioned similarity scaling. The difference in these and the results from the present model is seen to be very small because of the fact that the maximum is predicted along the mean

wind. For z=0, the two would be the same because the slender plume approximation and solution (33) are identical at y=0 and z=0. It means along wind diffusion is not so important for ground-level centerline concentrations, in particular, in the case of ground-level emissions. Theoretically the downwind diffusion is important away from the plume centerline [18,7,6].

Table 2. Peak values of tracer concentration (ppt) observed and predicted by various cases at 50m downwind of the source.

Run No.	Observed	Present model	Sharan Model	Similarity Present model	Gaussian Similarity Sharan	Briggs	Arya (1995a)
1	832	267	123	864	133	621	68
2	1068	184	67	214	76	1050	48
6	1101	185	90	564	97	420	51
7	248	167	77	545	81	421	44
8	1282	487.4	333	1329	354	1200	186
11	616	219	90	409	91	749	55
12	759	277	125	890	139	556	69
13	1060	380.6	164	786.4	178	911	95

Table 3. Peak values of tracer concentration (ppt) observed and predicted by various cases at 100m downwind of the source.

Run No.	Observed	Present model	Sharan model	Similarity Present model	Gaussian Similarity Sharan	Briggs	Arya (1995a)
1	345	266.8	38	376	41	192	22
2	460	125	17	142	19	261	14
6	176	177	22	352	24	105	14
7	288	165	19	364	20	105	12
8	345	254	84	340	88	300	51
11	162	167	23	262	23	188	15
12	222	163.6	31	250	35	139	19
13	215	323	41	245	44	228	27

For the sake of comparison, Tables 2. and 3. include the results from the simple Gaussian plume model with dispersion parameters estimated from Briggs' analytical expressions for urban terrain [19]. The results indicate that the simulated peaks from present model and Gaussian plume model using the similarity present model and Briggs' parameterizations for sigma's compare well with the observed peaks.

The results from this approach show that there are good agreement between prediction and observed peak concentrations on both 50 and 100m arcs (Tables 2. and 3.).

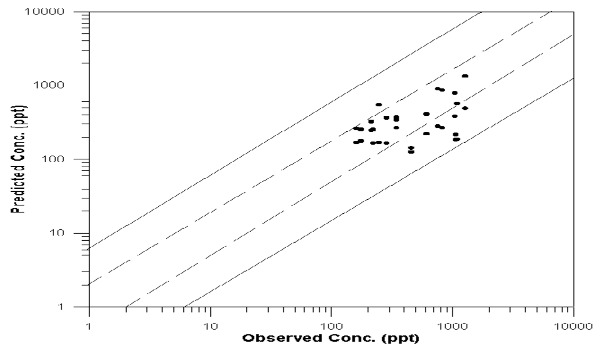

Figure 1. Scatter diagram of the peak model predictions for convective cases and the corresponding observations. Dashed lines indicate a factor of two and solid a factor of six.

Figure 1. shows the scatter diagram of the peak of the predicted and observed concentrations from all the test runs. It may be seen that the predicted concentration are inside factor of two with the observed concentrations.

CONCLUSION

This paper has simply brought to light various possible ways of accounting for the point source in the mathematical description of air pollution dispersion problems. A mathematical procedure has been provided in order to obtain alternative formulations of the problem using transform axis**,** cosine, and Fourier transform, leading to the same solution of the diffusion equation.

The solution for the ground source has been used to compare with observed concentrations data for SF_6 tracer experiments in low wind and unstable conditions at IIT Delhi sports ground. The results indicate that the simulated peaks from present model and Gaussian plume model using the similarity present model and Briggs' parameterizations for sigma's compare well with the observed peaks.

This, essentially, would help in understanding the problem better and extending it to more general situations.

REFERENCES

[1] Pasquill, F.A., and Smith, F.B., *Atmospheric Diffusion*, Ellis Horword Chichester, UK, 3rd Edn., 1983.

[2] Seinfeld, J.H., *Atmospheric Chemistry and Physics of Air Pollution*, Wiley, New York, 1986.

[3] Zanneti, P., *Air Pollution Modeling, Computational Mechanics*, Puplications, Southampton, 1990.

[4] Turner, D.B., "*Workbook of Atmospheric Dispersion Estimates*", Lewis Publishers CRC Press, Boca Raton, 1994.

[5] Csanady, G.T., *Turbulent Diffusion in the Environment*, Reidel Dordrecht, Holland, 1973.

[6] Arya, S.P.S., *Journal of Applied Meteorology* **30**, 57(1995).

[7] Sharan, M., Singh, M.P., and Yadav, A.K., *Atmospheric Environment* **30**, 1137(1996).

[8] Sharan, M., and Yadav, A. K., *Atmospheric Environment* **32**, 3481(1998); Essa, Khaled S.M. and El-Otaify, Maha S., *Meteorol Atmos Phys*, **96**, 265(2007).

[9] Sharan, M., Yadav, A.K., Singh, M.P., Agarwal, P., and Nigam S., *Atmospheric Environment* **30**, 1209(1995).

[10] Wortmann, S., Vilhena, M.T., Moreira, D.M. and Buske, D.,*Atmospheric Environment* **39**, 2171(2005).

[11] Sharan, M., Yadav, A.K., Singh, M.P. and Gupta S., *Atmospheric Environment* **33**, 1327(1999).

[12] Messiah, A., *Quantum Mechanics*, North Holland Publishing Comoany, Amsterdam, 1961.

[13] Morse, P.M. and Feshbach, H., *Methods of Theoretical Physics,* Mc Graw-Hill book Company, Inc., New York Toronto London, 1953, Par.I.

[14] Campbell, G.A. and Foster, R.M., *Fourier Integrals Practical Applications*, Van Nostrand, New York (1957).

[15] Gradshteyn, I.S. and Ryzhik, I.M., *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[16] Singh, M.P., Agarwal, P., Nigam, S. and Amita, Gulati, "Tracer Experiments", *Tech. Report*, CAS, IIT Delhi, 1991.

[17] Deardoff, J.W., in *Proceedings of the Fourth Joint Conference on Applications of Air Pollution Meorology*, Portland, Amer. Met. Society, Boston, Massachusetts (1984).

[18] Zannetti, P., "*In Encyclopedia of Environmental Control Technology*" (edited by Cheremisinoff P. N.) Gulf Publishing Company, Houston, 1989, vol 2, p.181.

المعالجة التحليلية المعممة لشدة المصدر فى ‏Dـ معادلة الانتشار

خالد صادق محمد عيسى و *مها سلامة العطيفى*

قسم الرياضة و الطبيعة النظرية , مركز البحوث النووية , هيئة الطاقة الذرية ,رقم بريدي

acgei , مصر

تم دراسة شدة انبعاث المصدر تحت شروط حدية مختلفة لمعادلة الانتشار وتوصلنا إلى ثلاثة أشكال مختلفة ومتكافئة الحلول فى نفس الوقت لمعادلة الانتشار. تم حل معادلة الانتشار ذات الانتشار الدوامى الثابت فى الأشكال الثلاثة السابقة وذلك باستخدام تحويل الإحداثيات الكارتيزية, تحويلات جيب التمام , وتحويلات فوريير وعليه أمكن الحصول على ثلاثة حلول متكافئة لمعادلة الانتشار. تم مقارنة بين التركيزات المحسوبة على سطح الأرض لمركب سادس كبريتيد الفلوريد عندما تكون سرعة الرياح صغيرة تحت شروط غير مستقرة مع التركيزات المقاسة على موقع IIT دلهى- الهند وتوصلنا أنه يوجد توافق جيد بين التركيزات المحسوبة والمقاس