CONTINUOUS STOCHASTIC RADIATIVE TRANSFER WITH RAYLEIGH SCATTERING IN SEMI-INFINITE ATMOSPHERIC MEDIA

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ABSTRACT

The radiative transfer problem in a semi-infinite stochastic atmospheric medium with Rayleigh scattering is studied. The extinction function (cross section) of the medium is assumed to be a continuous random function of position, with fluctuations about the mean taken as Gaussian distributed. The joint probability distribution function of these Gaussian random variables is used to calculate the ensemble-averaged quantities, such as radiant energy and net flux, for an arbitrary correlation function. The deterministic solution of the considered problem is obtained at first. Then the solution is averaged using Gaussian joint probability distribution function. A modified Gaussian probability distribution function is also used to average the solution. Numerical results are given for the sake of comparison.

Keywords: *Stochastic radiative transfer, Rayleigh scattering, Gaussian statistics, Atmospheric medium*

INTRODUCTION

One of the most significant problems that presently encountered in the field of transport phenomenon concerns how best to describe the transfer through a medium that is highly inhomogeneous over a vast range of length scales. Due to the importance of this task, the formulating of radiative transfer descriptions in a stochastic media has a great interest in the present time. By a stochastic we mean that the properties of the background material of the medium, with which the radiation interacts, are only known in a statistical sense. Our use of the word, stochastic means that the properties of the medium are either specified or can be computed in a statistical sense.

In the traditional statistical description of the radiative transfer characteristics of inhomogeneous atmospheric media (e.g., fog, clouds and snow), the stochasticity arises on the atomic scale due to the random location of scattering centers. On averaging the radiation propagation over small lengths and time scales, the celebrated Boltzmann equation is obtained. The latter is a deterministic equation for the expected (or mean) density of particles in a unit phase space volume. Fluctuations about this mean density occur on lengths and time scales much shorter than the mean free paths and mean times between collisions. Their description requires a higher order treatment. Studies based on traditional single-view instruments, which lack the ability of capturing the instantaneous distribution of angular radiances, can offer only statistical comparisons with generic reflectance models composited from several different cloud scenes [1], [2].

In most applications involving linear transport processes, however, the magnitude of fluctuations is generally small enough so that a description based on the expected number suffices, e.g., as in the traditional transport equation for neutrons, photons, charged particles, etc. [3]. Thus one deals with a purely deterministic equation characterized by microscopic parameters such as cross sections and macroscopic parameters such as the environmental density and temperature. The radiative transfer through the medium, which interact with a background material, but not with themselves, is described in some generality by this transport equation. While it is algebraically complex, it has a very simple physical content. It is simply the mathematical statement of particles conservation in phase space. Much of the fundamental understanding of physics contained in this equation, and the development of elegant mathematical methods to describe this physics, was pioneered by Chandrasekhar [3], Case and Zweifel [4] and Pomraning [5].

The stochastic transport class of problems arises when the environmental properties of the background material of the medium are random functions of position and time that in scales are of the order of, or longer than, mean free paths. The scientific texts treating the stochastic transport can essentially be divided into two major classes, in accordance with the definition of the randomness of background material properties. The first class deals with two-phase random media (discrete stochastic media). Models of this kind were elaborated in detail by many authors (cf. e.g. [6], [7] and the references cited therein). In this case the most extensive results have been obtained for the so-called Markovian mixtures.

The second class of stochastic media is related to the theory of fluctuations (continuous stochastic media). The advection and dispersion of a passive scalar (concentration, temperature, etc.) in a turbulent fluid is a classic example of this phenomenon, where the fluid velocity is assumed to be a random function with known statistical properties [8]. An application that motivated this work is the transport of neutral atoms in tokamak plasmas [9]. Such plasmas are known to be turbulent, and to a good approximation, the plasma density variations are well represented by a Gaussian stochastic process. Also, the model of Gaussian fluctuations has been used to investigate the effect on the dose due to a charged-particle beam of random fluctuations in the material density [10].

In models belonging to the second class one defines, as a rule, the properties of the medium as a Gaussian random function. Here, we can quote some papers by Prinja as the leading author [9]-[14]. In former papers [15]-[17], we had also dealt with Gaussian random media. The statistical moments of the stochastic quantities have been calculated with Gaussian statistics.

Throughout this paper we extend the previous Gaussian model [15]-[17], to the case of transport problem with Rayleigh scattering in a semi-infinite atmospheric medium. Rayleigh scattering law can be applied when the volume of constituent particles of mixture are much smaller than the incident radiation wavelength. For particles whose sizes are comparable to or larger than wavelength, the scattering is customarily referred to as Mie scattering. So, this problem of Rayleigh scattering gives an interesting application of radiative transfer in astrophysical setting. The advantage of Gaussian model is that it is easy to characterize, requiring only the mean, variance, and two-point correlation function to completely specify the distribution of random field. However, a weakness of the Gaussian mode is that the Gaussian random variables that are physically constrained to be positive can in fact, from mathematical point of view, take on negative values albeit with exponentially decreasing probability. This can potentially give cause for concern when ensemble averages are considered, especially when the fluctuation amplitude is large. However, a modified Gaussian probability density function [16], can be used here to overcome this defect.

In this paper, we obtain the deterministic solution of the radiative transfer equation with Rayleigh scattering (quadratic scattering) in a semi infinite medium. The reflectivity at the boundary as well as the radiant energy and the net heat flux are calculated. The medium is assumed to have specular-reflecting boundary with angular-dependent externally-incident flux. Then we clarify the stochasticity of the extinction function, $\sigma(z)$, its mean, variance, and autocorrelation function. Gaussian probability and modified Gaussian probability density functions are used to obtain the average radiant energy and the average net heat flux. Numerical results are obtained for different values of specular reflectivity and single scattering albedo.

PROBLEM FORMULATION

The monoenergetic transport equation with anisotropic scattering takes the form [6]

$$\left[\mu\frac{\partial}{\partial z}+\sigma(z)\right]I(z,\mu)=\frac{\sigma_s(z)}{2}\int_{-1}^{1}P(\mu,\mu')I(z,\mu')\,d\mu' \tag{1}$$

$$0\le z<\infty,\quad -1\le\mu\le 1$$

where $I(z, \mu)$ is the radiation intensity, with z, and μ representing the spatial and angular variables, respectively. The quantity $\sigma(z)$ is the extinction function and $\sigma_s(z)$ is the scattering cross-section.

The anisotropic scattering phase function, $P(\mu,\mu')$ is given by [18]

$$P(\mu,\mu')=\sum_{n=0}^{\infty}a_nP_n(\mu)P_n(\mu') \tag{2}$$

with $P_n(\mu)$ is the n^{th} Legendre polynomial functions.

Here a_n can be called the anisotropy scattering coefficients with $a_0 = 1$. The first term of this expansion is called isotropic scattering ($a_1 = a_2 = 0$). The probability of particle scattering is equal for all directions in isotropic scattering. The second term in this expansion corresponds to the linearly anisotropic scattering ($a_1 \neq 0$, $a_2 = 0$). The third term corresponds to the Rayleigh scattering (quadratic scattering $a_1 = 0$, $a_2 = 0.5$).

It is convenient to write Eq.(1) in terms of the optical depth space variable [6]

$$\tau(z)=\int_0^z\sigma(z)dz \tag{3}$$

to become

$$\left(\mu\frac{\partial}{\partial\tau}+1\right)\Psi(\tau,\mu)=\frac{\omega}{2}\int_{-1}^{1}P(\mu,\mu')\Psi(\tau,\mu')\,d\mu' \tag{4}$$

where

$$\Psi(\tau,\mu)\equiv I(z,\mu)\quad\text{and}\quad\omega=\sigma_s/\sigma \tag{5}$$

Equation (4) is assumed to subject to the boundary conditions

$$\Psi(0,\mu)=\Lambda(\mu)+\rho^s\Psi(0,-\mu)\ , \tag{6}$$

$$\lim_{\tau\to\infty}\Psi(\tau,-\mu)=0 \tag{7}$$

where $\Lambda(\mu)$ is the angular-dependent externally-incident flux, and ρ^s is the specular reflectivity of this boundary.

For Rayleigh scattering phase function [7], Eq.(2) becomes

$$P(\mu,\mu')=1+\frac{1}{2}P_2(\mu)P_2(\mu') \tag{8}$$

Using Eq.(8) into Eq.(4), we have

$$\left(\mu\frac{\partial}{\partial\tau}+1\right)\Psi(\tau,\mu)=\frac{\omega}{2}\int_{-1}^{1}\left(1+\frac{1}{2}P_2(\mu)P_2(\mu')\right)\Psi(\tau,\mu')d\mu' \quad \textbf{(9)}$$

The transport equation of the type given by Eqs.(9) admits separable exponential solutions of the form

$$\Psi(\tau,\mu)=A\phi(\mu,\nu)\exp(-\nu\tau) \quad \textbf{(10)}$$

where $\phi(\mu,\nu)$ is a normalized function and A is the normalization constant to be determined.

Using of Eq.(10) in Eq.(9) yields

$$(1-\nu\mu)\phi(\mu,\nu)=\frac{\omega}{2}\left[H_0(\nu)+\frac{1}{2}P_2(\mu)H_2(\nu)\right] \quad \textbf{(11)}$$

with

$$H_j(\nu)=\int_{-1}^{1}P_j(\mu)\phi(\mu,\nu)d\mu \quad , \qquad j=0,2 \quad \textbf{(12)}$$

Integrating Eq.(11) over $\mu \in [-1, 1]$ gives

$$(1-\omega)H_0(\nu)-\nu H_1(\nu)=0 \quad \textbf{(13)}$$

Multiplying Eq.(11) by μ and integrate over $\mu \in [-1, 1]$, one has

$$H_1(\nu)-\frac{\nu}{3}\left[H_0(\nu)+2H_2(\nu)\right]=0 \quad \textbf{(14)}$$

Dividing Eq.(11) by $(1 - \nu\mu)$ and integrate over $\mu \in [-1, 1]$ we get

$$\left[1-\frac{\omega}{\nu}q_0\left(\frac{1}{\nu}\right)\right]H_0(\nu)-\frac{\omega}{2\nu}q_2\left(\frac{1}{\nu}\right)H_2(\nu)=0 \quad \textbf{(15)}$$

where we have defined the n^{th} order Legendre function of second kind, $q_n(y)$, as

$$q_n(y)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(\mu)}{y-\mu}d\mu \quad \textbf{(16)}$$

with

$$q_0\left(\frac{1}{\nu}\right)=\frac{1}{2}\ln\left(\frac{1+\nu}{1-\nu}\right) \quad \text{and} \quad q_2\left(\frac{1}{\nu}\right)=\frac{1}{4}\left(\frac{3}{\nu^2}-1\right)\ln\left(\frac{1+\nu}{1-\nu}\right)-\frac{3}{2\nu} \quad \textbf{(17)}$$

Eqs.(13)-(15) constitute four linear homogeneous algebraic equations for the three unknowns H_0, H_1 and H_2. The corresponding vanishing of the coefficient determinant gives the transcendental equation (characteristic equation) satisfied by ν, as

$$\nu=\omega\left\{q_0\left(\frac{1}{\nu}\right)+\frac{1}{4}\left[\frac{3}{\nu^2}(1-\omega)\right]q_2\left(\frac{1}{\nu}\right)\right\} \quad \textbf{(18)}$$

Now for normalized $\phi(\mu,\nu)$ [i.e. $H_0(\nu) = 1$], we have

$$H_1(\nu)=\frac{1}{\nu}(1-\omega) \quad \text{and} \quad H_2(\nu)=\frac{1}{2}\left[1-\frac{3}{\nu^2}(1-\omega)\right] \quad \textbf{(19)}$$

Using Eq.(19) in Eq.(11) we obtain

$$\phi(\mu,\nu)=\frac{\omega}{2(1-\nu\mu)}\left\{1+\frac{1}{4}\left[1-\frac{3}{\nu^2}(1-\omega)\right]P_2(\mu)\right\} \quad \textbf{(20)}$$

and hence we obtain the solution in the form

$$\Psi(\tau,\mu)=\frac{\omega}{2}A\left\{1+\frac{1}{4}\left[1-\frac{3}{\nu^2}(1-\omega)\right]P_2(\mu)\right\}\frac{\exp(-\nu\tau)}{(1-\nu\mu)} \quad \textbf{(21)}$$

The constant A can be determined by introducing a weight function $W(\mu)$ in order to force the boundary conditions Eq.(6) to be fulfilled, as

$$\int_0^1 d\mu W(\mu)\left[\Psi(0,\mu)-\Lambda(\mu)-\rho^s\Psi(0,-\mu)\right]=0 \quad \textbf{(22)}$$

this can give

$$A=\left(\frac{2}{\omega}\right)\frac{J_0}{J_+-\rho^s J_-} \quad \textbf{(23)}$$

where

$$J_0=\int_0^1 W(\mu)\Lambda(\mu)d\mu \quad \text{and} \quad J_\pm=\int_0^1 W(\mu)\Gamma(\pm\mu)d\mu \quad \textbf{(24)}$$

with

$$\Gamma(\pm\mu)=\frac{1}{1\mp\nu\mu}\left\{1+\frac{1}{4}\left[1-\frac{3}{\nu^2}(1-\omega)\right]P_2(\mu)\right\} \quad \textbf{(25)}$$

The solution is, then given by

$$\Psi(\tau,\mu)=\frac{J_0\Gamma(\mu)}{J_+-\rho^s J_-}\exp(-\nu\tau) \quad \textbf{(26)}$$

Equation (26) represents the explicit analytical deterministic solution for the problem under consideration. Now, we can calculate the reflectivity at the boundary of the semi-infinite medium as $R=\int_0^1 \mu\Psi(0,-\mu)d\mu$, to give

$$R=\frac{J_0 J_r}{J_+-\rho^s J_-} \quad \text{where} \quad J_r=\int_0^1 \mu\Gamma(-\mu)d\mu \quad \textbf{(27)}$$

Further, we could calculate the radiant energy and the net heat flux of the propagating radiation, respectively, as

$$E(\tau)=\int_{-1}^1 \Psi(\tau,\mu)d\mu, \quad \text{and} \quad F(\tau)=\int_{-1}^1 \mu\Psi(\tau,\mu)d\mu \quad \textbf{(28)}$$

Using Eq.(26) in Eqs.(28), we get the deterministic values of $E(\tau)$ and $F(\tau)$ as

$$E(\tau)=\frac{J_0 J_e}{J_+-\rho^s J_-}\exp(-\nu\tau), \quad \text{and} \quad F(\tau)=\frac{J_0 J_f}{J_+-\rho^s J_-}\exp(-\nu\tau) \quad \textbf{(29)}$$

where

$$J_e=\int_{-1}^1 \Gamma(\mu)d\mu, \quad \text{and} \quad J_f=\int_{-1}^1 \mu\Gamma(\mu)d\mu \quad \textbf{(30)}$$

STATISTICAL ANALYSIS

In the usual (nonstochastic) application of the transport equation, Eq.(1), the extinction coefficient, $\sigma(z)$, and the scattering cross section, $\sigma_s(z)$, are known (deterministic) prescribed functions. The goal is simply to solve this equation with respect to the boundary conditions given by Eq.(6) for $I(z,\mu)$. In the stochastic setting $\sigma(z)$ and $\sigma_s(z)$ are only known in some probabilistic sense. That is, at each space point z there is some probability that each of these two quantities will assume certain values. Accordingly, we consider $\sigma(z)$ and $\sigma_s(z)$ to be random functions, and then $I(z,\mu)$ is also random. On transforming Eq.(1) from the geometrical space to the optical space, Eq.(3), the stochasticity of the problem is absorbed in the optical variable τ. It is also assumed here that the cross section is a random function of position such that the single scattering albedo $\omega = \sigma_s/\sigma$ is not random.

On treating the random function $\sigma(z)$, firstly, we assume that it is a statistically homogeneous random function. This means that if we replace all values of z_i by $z + z_i$, (i = 1,2, … , m), the average values of the product $\sigma_1(z_1)\ \sigma_2(z_2)\ \sigma_3(z_3) \ldots \sigma_m(z_m)$, (m = 2,3, ….) do not depend on the geometrical depth space variable z. Secondly, we exemplify $\sigma(z)$ by a Gaussian random function with a constant mean value $\bar{\sigma} = \langle \sigma(z) \rangle$ and a constant variance η_σ^2. From the probabilistic point of view, any Gaussian random function is defined completely if its mean, $\bar{\sigma}$, and its autocorrelation function, $W_\sigma(z_1,z_2)$, are defined. We define the autocorrelation of the random function $\sigma(z)$ as

$$W_\sigma(z_1,z_2) = \langle [\sigma_1(z_1) - \bar{\sigma}][\sigma_2(z_2) - \bar{\sigma}] \rangle \quad \textbf{(31)}$$

where z_1 and z_2 are arbitrary positions. For homogeneous statistics, $W_\sigma(z_1,z_2)$ depend on $|z_1 - z_2|$. It can be expressed as

$$W_\sigma(z_1,z_2) = W_\sigma(|z_1 - z_2|) = \eta_\sigma^2 B(|z_1 - z_2|) \quad \textbf{(32)}$$

where the variance η_σ^2 is given by

$$\eta_\sigma^2 = \left\langle [\sigma(z) - \bar{\sigma}]^2 \right\rangle > 0 \quad \textbf{(33)}$$

We assume that the function $W_\sigma(z_1,z_2)$ is positive for real values of z. Moreover, $W_\sigma(0)=\eta_\sigma^2$ and $W_\sigma(z) \to 0$ as $z \to \infty$. The shape function, B, describes the range over which the parameters fluctuations are correlated. It is typically characterized by correlation length ℓ such that $B \approx 0$ for $|z_1 - z_2| >> \ell$ and $B(0) = 1$. The form of B depends on the specific application, but in many cases, it is modeled by simple exponentials. However, the shape function, $B(|z_1 - z_2|)$ is a deterministic function and must be given in advance. Important and common employed models are the exponentials [10], [15]

$$B(|z|) = \exp\left(-\frac{|z|}{\ell}\right) \quad \textbf{(34)}$$

or

$$B(|z|) = \exp\left(-\frac{z^2}{\ell^2}\right) \quad \textbf{(35)}$$

If σ_1 and σ_2 are two Gaussian random functions, then $\sigma_1+\sigma_2$ is also Gaussian random function. Generalizing this statement, we may say that, if $\sigma(z)$ is a Gaussian random function, then the optical depth space variable τ is also Gaussian. This fact follows from the linearity of the formula (3). The mean value of τ is

$$\bar{\tau} = \langle \tau \rangle = \int_0^z dz' \langle \sigma(z') \rangle = \bar{\sigma} z \quad \textbf{(36)}$$

The variance η^2 of τ depends on the form of the function $W(|z|)$ and can be written as

$$\eta^2 = \left\langle (\tau - \bar{\tau})^2 \right\rangle = \int_0^z dz_1 \int_0^z dz_2 \langle [\sigma(z_1) - \bar{\sigma}][\sigma(z_2) - \bar{\sigma}] \rangle = \int_0^z dz_1 \int_0^z dz_2 W_\sigma(|z_1 - z_2|) \quad \textbf{(37)}$$

We refrain from calculating the parameter η and take simply both $\tau > 0$ and $\eta > 0$ as given constants. We require that $0 < \eta_\sigma << \bar{\tau}$.

However, the randomness of the function $\sigma(z)$ is defined by three parameters: $\bar{\sigma}$, η_σ and the correlation length ℓ. If $\eta_\sigma \to 0^+$, our stochastic problem goes over into the deterministic one. The mean value $\bar{\tau}$ of the optical variable τ is dimensionless quantity proportional to the factual length z, Eq.(36). It is suitable to introduce a positive dimensionless parameter

$$\beta = \eta_\sigma \sqrt{\frac{\ell}{\bar{\sigma}}} \quad \textbf{(38)}$$

The constants $\bar{\sigma}$, η_σ and ℓ (and then also the parameter β) are statistical characteristics of the random media. The value of β may be more or less arbitrary, serving for the characterization of the random medium under consideration. To estimate the value of the quantity η according to formula (37), we have to choose the shape of the autocorrelation function $W(|z|)$. Let us take $W(|z|)$ in the exponential form [10], [15] (see formulae (34))

$$W(|z|) = \eta_\sigma^2 \exp\left(-\frac{|z|}{\ell}\right) \quad \textbf{(39)}$$

Using Eq. (39) in Eq.(37), assuming that $\ell << z$, we get

$$\eta^2 \approx 2\eta_\sigma^2 \ell z \approx 2\beta^2 \bar{\sigma} z \quad \textbf{(40)}$$

If $W(|z|)$ was chosen in a form differing from the simple exponential, or if the correlation length ℓ was defined with a factor different from unity, formula (40) would still be valid, although with a factor different from 2. For the sake of simplicity, we will take formula (40) as approximately correct even in the case of small values of z.

THE AVERRAGE SOLUTION

In the previous treatment we have assumed that τ is a Gaussian real random variable and $\bar{\tau} = \langle \tau \rangle \geq 0$. In defining τ as a Gaussian random variable, we have to assume that values of τ may span the whole real axis, including the negative values. We have to require the existence of the variance $\eta^2 > 0$. The Gaussian probability density of τ is then

$$P^G(\tau, \eta^2) = \frac{1}{\sqrt{2\pi\eta^2}} \exp\left(-\frac{(\tau - \bar{\tau})^2}{2\eta^2}\right) \quad \textbf{(41)}$$

It is easy to verify that the averaged value of $\exp[-k(\tau - \bar{\tau})]$ with a constant k, is

$$Z^G(k, \eta^2) = \langle \exp[-k(\tau - \bar{\tau})] \rangle = \int_{-\infty}^{\infty} d\tau \, e^{-k(\tau - \bar{\tau})} P^G(\tau, \eta^2) = \exp\left(\frac{k^2\eta^2}{2}\right) \quad \textbf{(42)}$$

which represents the characteristic function of the Gaussian distribution.

Often it may happen that τ is a physical quantity for which we have to respect a constraint. Throughout this paper, we assume that $\tau > 0$. If $0 < \eta << \bar{\tau}$, the probabilistic distribution of τ, excluding its negative values, may still very close to the Gaussian. Indeed, employing the unit step function $\Theta(\tau) = 1$ if $\tau > 0$ and $\Theta(\tau) = 0$ if $\tau < 0$, we may define a modified Gaussian probability density

$$P_M^G(\tau, \eta^2) = \frac{\Theta(\tau) P^G(\tau, \eta^2)}{\int_0^\infty d\tau' P^G(\tau', \eta^2)} \quad \textbf{(43)}$$

The explicit form of the modified Gaussian probability density, $P_M^G(\tau, \eta^2)$, is given like we have shown in [15] and [16] as

$$P_M^G(\tau, \eta^2) = \frac{N(\bar{\tau}, \eta^2)}{\sqrt{2\pi\eta^2}} \Theta(\tau) \exp\left(-\frac{(\tau - \bar{\tau})^2}{2\eta^2}\right) \quad \textbf{(44)}$$

The averaged value of $\exp\left[-k\left(\tau-\bar{\tau}\right)\right]$ with the modified Gaussian probability density $P_M^G\left(\tau,\eta^2\right)$ is given by the integral

$$Z_M^G\left(k,\eta^2\right)=\int_{-\infty}^{\infty}d\tau\exp\left[-k\left(\tau-\bar{\tau}\right)\right]P_M^G\left(\tau,\eta^2\right) \tag{45}$$

After performing the integral on the RHS of Eq. (45), we get

$$Z_M^G\left(k,\eta^2\right)=N\left(\bar{\tau},\eta^2\right)\left\{1-\frac{1}{2}erfc\left(\frac{\left(\bar{\tau}-k\eta^2\right)}{\sqrt{2\eta^2}}\right)\right\}\exp\left(\frac{k^2\eta^2}{2}\right) \tag{46}$$

where

$$N\left(\bar{\tau},\eta^2\right)=\left[1-\frac{1}{2}erfc\left(\bar{\tau}/\sqrt{2\eta^2}\right)\right]^{-1} \tag{47}$$

and the complementary error function is defined as

$$erfc\left(u\right)=\frac{2}{\sqrt{\pi}}\int_u^{\infty}dt\exp\left(-t^2\right) \tag{48}$$

It is clear from Eq.(27) that the reflectivity of the semi-infinite medium is independent of τ, i.e., it is independent of the stochasticity of the medium. So, formula (27) of the reflectivity has no parameters concern with the randomness of the stochastic medium.

Now, we can use the Gaussian probability density function (41) and Eq.(42) to evaluate the average values of the exponentials that appear in Eqs. (29) as

$$E^G\left(\nu,\bar{\tau}\right)=\left\langle\exp\left(-\nu\tau\right)\right\rangle_G=\exp\left(\frac{\nu^2\eta^2}{2}-\nu\bar{\tau}\right) \tag{49}$$

In a similar way, we can use the modified Gaussian probability density function (44), and Eq.(46) to average the exponentials that appear in Eqs. (29) as

$$E^{MG}\left(\nu,\bar{\tau}\right)=\left\langle\exp\left(-\nu\tau\right)\right\rangle_{MG}=N\left(\bar{\tau},\eta^2\right)\left\{1-\frac{1}{2}erfc\left(\frac{\left(\bar{\tau}-\nu\eta^2\right)}{\sqrt{2\eta^2}}\right)\right\}\exp\left(\frac{\nu^2\eta^2}{2}-\nu\bar{\tau}\right) \tag{50}$$

In terms of the statistical parameter β given by Eq.(38), Eqs.(49) and (50) can be rewritten as

$$E^G\left(\nu,\bar{\tau}\right)=\left\langle\exp\left(-\nu\tau\right)\right\rangle_G=\exp\left[\left(\nu^2\beta^2-\nu\right)\bar{\tau}\right] \tag{51}$$

and

$$E^{MG}\left(\nu,\bar{\tau}\right)=\left\langle\exp\left(-\nu\tau\right)\right\rangle_{MG}=N\left(\bar{\tau},\beta\right)\left\{1-\frac{1}{2}erfc\left(\frac{\left(1-2\nu\beta^2\right)\sqrt{\bar{\tau}}}{2\beta}\right)\right\}\exp\left[\left(\nu\beta^2-1\right)\nu\bar{\tau}\right] \tag{52}$$

where

$$N\left(\bar{\tau},\beta\right)=\left[1-\frac{1}{2}erfc\left(\sqrt{\bar{\tau}}/2\beta\right)\right]^{-1} \tag{53}$$

and $\bar{\tau}=\bar{\sigma}z$. It can be seen that the case of the statistical parameter $\beta = 0$ corresponds to the deterministic case. That is, there are no (or negligible) fluctuations in the medium. Therefore, increasing the value of β means that the randomness and fluctuations are increased. It is clear from Eqs.(51) and (52) that if we put $\beta = 0$, these equations are reduced to the deterministic case.

Hence, the average radiant energy is written as

$$\left\langle E\left(\bar{\tau}\right)\right\rangle_G=\frac{J_0J_e}{J_+-\rho^sJ_-}E^G\left(\nu,\bar{\tau}\right),\quad\text{and}\quad\left\langle E\left(\bar{\tau}\right)\right\rangle_{MG}=\frac{J_0J_e}{J_+-\rho^sJ_-}E^{MG}\left(\nu,\bar{\tau}\right) \tag{54}$$

and the average net heat flux is

$$\left\langle F\left(\bar{\tau}\right)\right\rangle_G = \frac{J_0 J_f}{J_+ - \rho^s J_-} E^G\left(\nu,\bar{\tau}\right), \quad \text{and} \quad \left\langle F\left(\bar{\tau}\right)\right\rangle_{MG} = \frac{J_0 J_f}{J_+ - \rho^s J_-} E^{MG}\left(\nu,\bar{\tau}\right) \qquad \textbf{(55)}$$

NUMERICAL RESULTS

In this section we present some numerical calculations for the reflectivity, *R*, the average radiant energy, $\left\langle E\left(\bar{\tau}\right)\right\rangle$, and the average net flux, $\left\langle F\left(\bar{\tau}\right)\right\rangle$. The externally-incident flux $\Lambda(\mu)$ is assumed to have the form

$$\Lambda\left(\mu\right) = \mu^n, \quad n = 0,1,2,\cdots \qquad \textbf{(56)}$$

For the sake of comparison, we use three different weight functions. The used weight functions have the special forms [7], [19], [20]

$$W_1\left(\mu\right) = \mu, \quad W_2\left(\mu\right) = \frac{\sqrt{3}}{2}\mu\left(1+\frac{3}{2}\mu\right) \quad \text{and} \quad W_3\left(\mu\right) = \mu\Psi^+\left(0,\mu\right) = \mu\Psi\left(0,-\mu\right) \qquad \textbf{(57)}$$

On the other side, we present, graphically, the numerical results of $\left\langle E\left(\bar{\tau}\right)\right\rangle$ and $\left\langle F\left(\bar{\tau}\right)\right\rangle$ as a functions of the optical variable $\bar{\tau} = \bar{\sigma} z$ for certain values of the statistical parameter β.

Table (1) gives the reflectivity *R* for three different groups of *n* and ρ^s. The comparison shows good agreement between the results calculated by the different forms of the weight function. This table shows that *R* increases as ρ^s and ω increase and decreases as *n* increases. Furthermore, we present, graphically, the numerical results of the average radiant energy $\left\langle E\left(\bar{\tau}\right)\right\rangle$ and the average net heat flux $\left\langle F\left(\bar{\tau}\right)\right\rangle$ as a functions of the average optical depth $\bar{\tau}$ inside the medium. The Gaussian distribution function and the modified one are used with certain values for the statistical parameter β.

Table (1) The Reflectivity R

ω	$W_1(\mu)$	$W_2(\mu)$	$W_3(\mu)$
(a) n= 0 and ρ^s = 0. 5			
0.2	0.031846	0.026892	0.035553
0.4	0.061501	0.054150	0.066731
0.6	0.107458	0.097983	0.113940
0.8	0.201704	0.189836	0.209548
0.9	0.309053	0.295864	0.317608
(b) n= 1 and ρ^s = 0. 5			
0.2	0.021231	0.019048	0.022797
0.4	0.041001	0.038356	0.042789
0.6	0.071639	0.069404	0.073060
0.8	0.134469	0.134467	0.134365
0.9	0.206036	0.209570	0.203655
(c) n= 1 and ρ^s = 0. 25			
0.2	0.020898	0.018810	0.022385
0.4	0.039777	0.037390	0.041370
0.6	0.067986	0.066274	0.069040
0.8	0.122150	0.123086	0.121439
0.9	0.178459	0.183014	0.175463

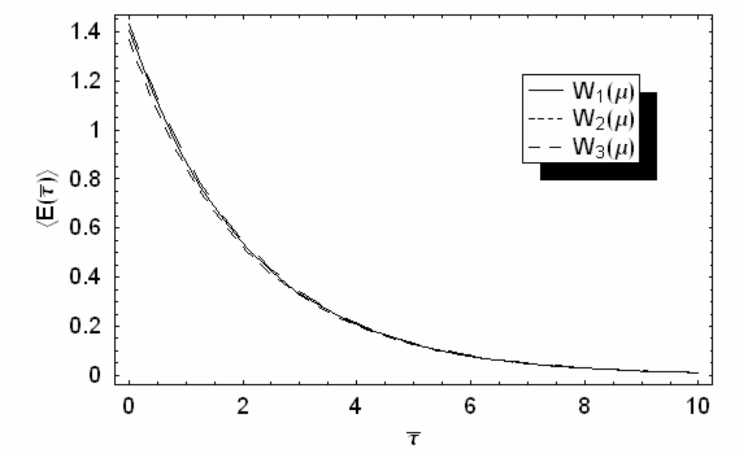
Figures Captions

1. Figures (1) give the data of <u>(a)</u> the average radiant energy $\langle E(\bar{\tau}) \rangle$ and <u>(b)</u> the average net heat flux $\langle F(\bar{\tau}) \rangle$ versus the average distance $\bar{\tau}$ for $n = \rho^s = 0$, $\omega = 0.9$, and statistical parameter $\beta = 0.4$ using Gaussian probability density.

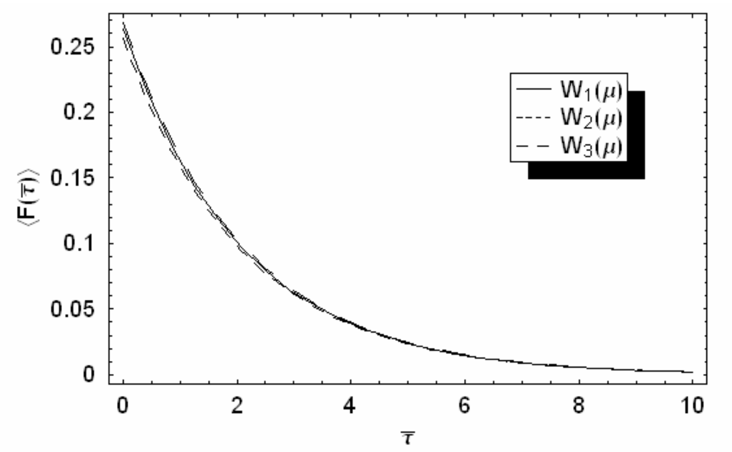
2. Figures (2) give the data of <u>(a)</u> the average radiant energy $\langle E(\bar{\tau}) \rangle$ and <u>(b)</u> the average net heat flux $\langle F(\bar{\tau}) \rangle$ versus the average distance $\bar{\tau}$ for $n = 1$, $\rho^s = 0.5$, $\omega = 0.9$, and statistical parameter $\beta = 0.3$ using the weight function $W_1(\mu)$.

3. Figures (3) give the data of <u>(a)</u> the average radiant energy $\langle E(\bar{\tau}) \rangle$ and <u>(b)</u> the average net heat flux $\langle F(\bar{\tau}) \rangle$ versus the average distance $\bar{\tau}$ for $n = 2$, $\rho^s = 0.25$, $\omega = 0.8$, for different values of statistical parameter β, using the weight function $W_1(\mu)$ and Gaussian probability density function.

(a) $\langle E(\bar{\tau}) \rangle$



(b) $\langle F(\bar{\tau}) \rangle$



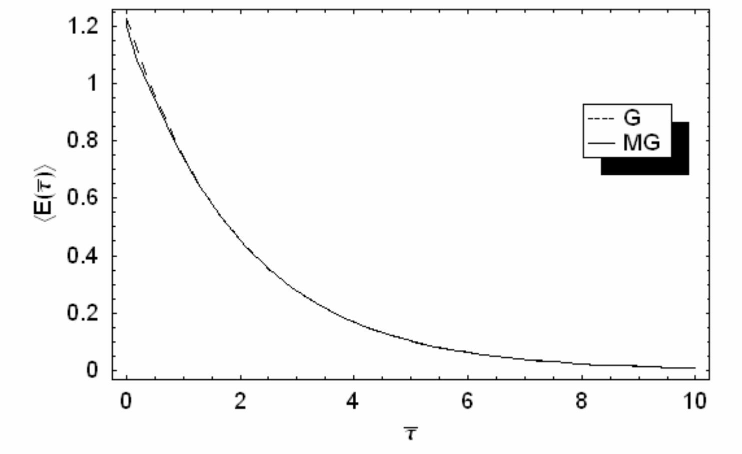
Figs.(1): <u>(a)</u> $\langle E(\bar{\tau}) \rangle$ <u>(b)</u> $\langle F(\bar{\tau}) \rangle$ for $n=\rho^s=0$, $\omega=0.9$ & $\beta = 0.4$

4. Figures (4) give the data of the average radiant energy $\langle E(\bar{\tau})\rangle$ versus $\bar{\tau}$ for unit incidence (n = 0) using Gaussian probability density function, for three groups of β and ρ^s, as

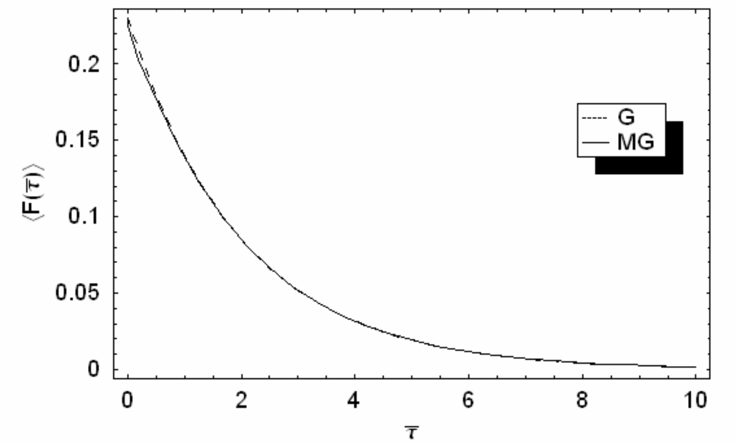
(a) β = 0.4 & ρ^s = 0.25, (b) β = 0.4 & ρ^s = 0.5, (c) β = 0.6 & ρ^s = 0.5.

5. Figures (5) show the results of the average net heat flux $\langle F(\bar{\tau})\rangle$ versus $\bar{\tau}$ for the same parameters as in Figures (4).

(a) $\langle E(\bar{\tau})\rangle$



(b) $\langle F(\bar{\tau})\rangle$

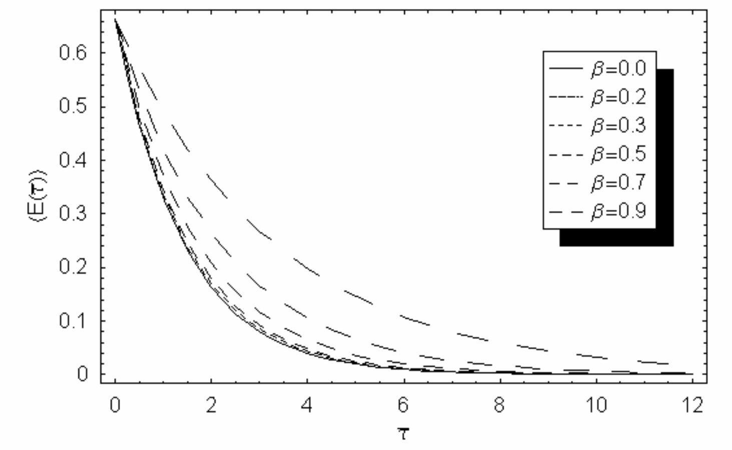


Figs.(2): (a) $\langle E(\bar{\tau})\rangle$ (b) $\langle F(\bar{\tau})\rangle$ for n= 1, ρ^s = 0.5, ω = 0.9 & β = 0.3

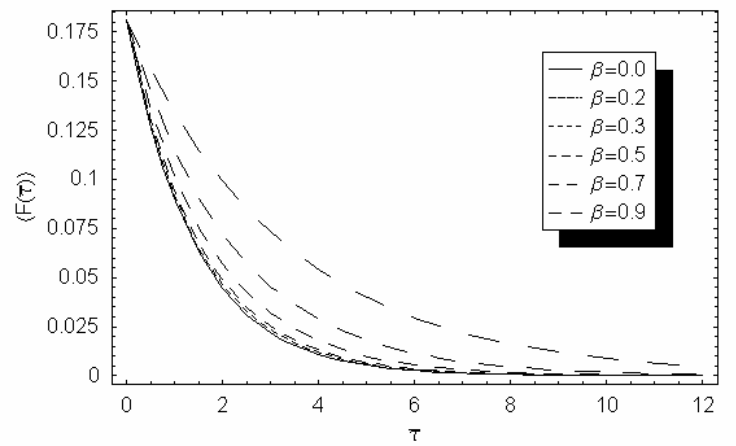
CONCLUSION

We have treated the transport problem through a continuous stochastic semi-infinite atmospheric medium in the presence of Rayleigh scattering. We try to study the effect of randomness in the properties of the medium on some physical quantities of interest like the reflectivity, the radiant energy and the net heat flux. The medium is assumed to have specular-reflecting boundary and angular-dependent externally-incident flux in the form μ^n. The deterministic solution of the considered problem is obtained at first. Then a Gaussian, as well as, a modified Gaussian probability density functions are used to average the solution over the medium fluctuations. We have used three different forms for the weight function that used to force the boundary conditions to be fulfilled. From the obtained numerical results, the following points can be concluded:

(a) $\langle E(\bar{\tau})\rangle$



(b) $\langle F(\bar{\tau})\rangle$



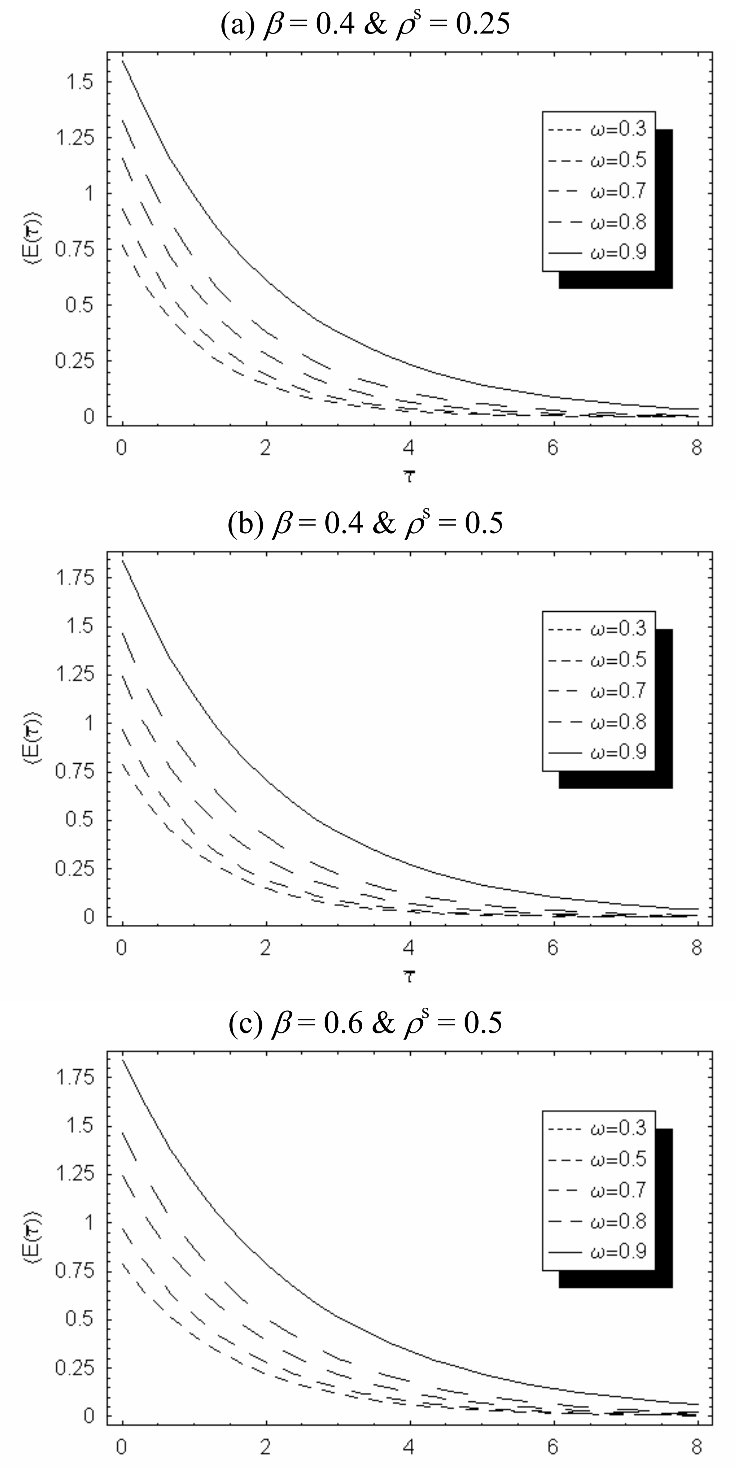
Figs.(3): (a) $\langle E(\bar{\tau})\rangle$ (b) $\langle F(\bar{\tau})\rangle$ for n= 2, $\rho^s = 0.25$ & $\omega = 0.8$

1- The reflectivity in the semi-infinite medium, regardless of the type of the scattering of the propagating radiation, is independent of the stochasticity of the medium.

2- The results calculated by the different three forms of the weight function are comparable with each other. So we have used only one weight function in the remaining graphs of $\langle E(\bar{\tau})\rangle$ and $\langle F(\bar{\tau})\rangle$. We select $W_1(\mu)$ because, as shown from table (1) and figures (1), it gives the mean results between the other data given by $W_2(\mu)$ and $W_3(\mu)$.

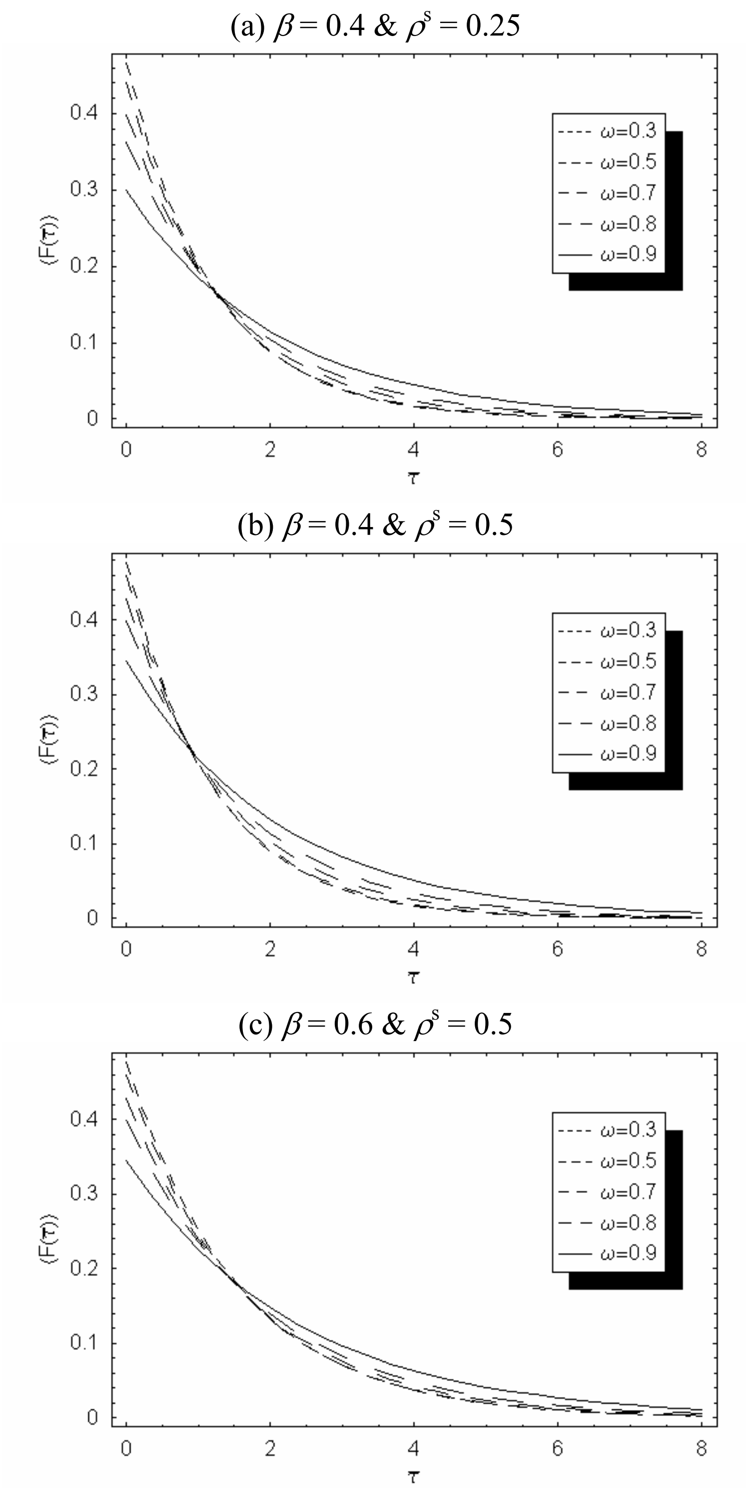
3- The averaged quantities ($\langle E(\bar{\tau})\rangle$ and $\langle F(\bar{\tau})\rangle$) calculated by using both Gaussian distribution function and the modified one are nearly the same especially for small statistical parameter β and/or higher values of $\bar{\tau}$. This because the Gaussian distribution and the modified are coincide with each other in this case (see Figs.(2)). So, the modified Gaussian is not suitable for very small $\bar{\tau}$ and/or for large β in case of semi-infinite medium.

4- For higher values of β, the decay of $\langle E(\bar{\tau})\rangle$ and $\langle F(\bar{\tau})\rangle$ decreases as a functions of the optical variable $\bar{\tau}$, as shown in Figs. (3). This is physically acceptable since the increasing of the fluctuations of the medium (i.e. increasing the randomness), the decreasing of the decay of $\langle E(\bar{\tau})\rangle$ and $\langle F(\bar{\tau})\rangle$.



Figs.(4): Average radiant energy $\langle E(\bar{\tau})\rangle$ for n = 0

5- It can be seen that the values of the average radiant energy $\langle E(\bar{\tau})\rangle$ increases as the scattering albedo, ω, and the specular reflectivity, ρ^s, of the boundary increase (see Figs.(4)). The situation is slightly different for the average net flux $\langle F(\bar{\tau})\rangle$. It increases as the specular reflectivity, ρ^s, of the boundary increases. However, $\langle F(\bar{\tau})\rangle$ decreases as ω increases until a critical point, after which $\langle F(\overline{\tau})\rangle$ increases as ω increases. This critical point gets a larger value of $\overline{\tau}$ as the specular reflectivity, ρ^s, becomes smaller and/or the statistical parameter becomes larger (see Figs.(5)). This is because of the effect of the surface of the medium.



Figs.(5): Average net heat flux $\langle F(\overline{\tau})\rangle$ for $n = 0$

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